2-4-2012

Absolute Value Equations

The absolute value of a number is defined by

$$|c| = \begin{cases} +c & \text{if } c \geq 0 \\ -c & \text{if } c < 0 \end{cases}.$$

Suppose you want to solve an absolute value equation of the form

$$|(\text{stuff})| = (\text{a number}).$$

Replace $|(\text{stuff})|$ with $\pm(\text{stuff})$, then take cases.

Example. Solve $|x| = 7$.

Replace $|x|$ with $\pm x$:

$$\begin{array}{ccc} & \pm x = 7 & \\ \swarrow & & \searrow \\ x = 7 & & -x = 7 \\ x = 7 & & x = -7 \end{array}$$

The solutions are $x = 7$ and $x = -7$.

It's often easier to put the "$\pm$" on the side with the number rather than the side with the variable, this way:

$$\begin{array}{ccc} & x = \pm 7 & \\ \swarrow & & \searrow \\ x = 7 & & x = -7 \end{array}$$

This is the approach I'll use in the examples that follow. □

Example. Solve $|x - 3| = 2$.

Remove the absolute values and put a "$\pm$" on the "2". Then take cases and solve.

$$\begin{array}{ccc} & x - 3 = \pm 2 & \\ \swarrow & & \searrow \\ x - 3 = 2 & & x - 3 = -2 \\ x = 5 & & x = 1 \end{array}$$

The solutions are $x = 5$ and $x = 1$. □

Example. Solve $|x + 4| = 6$.

Remove the absolute values and put a "$\pm$" on the "6". Then take cases and solve.

$$\begin{array}{ccc} & x + 4 = \pm 6 & \\ \swarrow & & \searrow \\ x + 4 = 6 & & x + 4 = -6 \\ x = 2 & & x = -10 \end{array}$$

The solutions are $x = 2$ and $x = -10$. □

Example. Solve $|2x - 3| = 5$.

Remove the absolute values and put a "$\pm$" on the "5". Then take cases and solve.

$$2x - 3 = \pm 5$$

$$\swarrow \qquad \searrow$$

$$\begin{array}{c} 2x - 3 = 5 \\ 2x = 8 \\ x = 4 \end{array} \qquad\qquad \begin{array}{c} 2x - 3 = -5 \\ 2x = -2 \\ x = -1 \end{array}$$

The solutions are $x = 4$ and $x = -1$. □

Example. Solve $|x - 7| = -8$.

If you work this problem like the others, you'll get two answers, but they won't be right.

The equation says an absolute value ($|x-7|$) is negative (-8). Since an absolute value can't be negative, the equation has no solutions. □

***Example.** Solve $|x| = |x - 4| + 9$.

Since I have two absolute value expressions, I'll go back to my original procedure: Remove the absolute values from an expression and put "$\pm$" on it. Doing so, I get

$$\pm x = \pm(x - 4) + 9.$$

Now I have 4 cases:

	x	$x - 4$
case 1	+	+
case 2	+	-
case 3	-	+
case 4	-	-

Case 1.

$$\begin{aligned} x &= (x - 4) + 9 \\ 0 &= 5 \end{aligned}$$

Since this is a contradiction, this case doesn't give any solutions.

Case 2.

$$\begin{aligned} x &= -(x - 4) + 9 \\ x &= -x + 4 + 9 \\ 2x &= 13 \\ x &= \frac{13}{2} \end{aligned}$$

Case 3.

$$\begin{aligned} -x &= (x - 4) + 9 \\ -x &= x + 5 \\ -2x &= 5 \\ x &= -\frac{5}{2} \end{aligned}$$

Case 4.

$$\begin{aligned} -x &= -(x-4)+9 \\ x &= -x+4+9 \\ 0 &= 13 \end{aligned}$$

Since this is a contradiction, this case doesn't give any solutions.

The solutions are $x = \dfrac{13}{2}$ and $x = -\dfrac{5}{2}$. □
