Solving Absolute Value Inequalities

This is a method for solving inequalities like

$$|ax+b|>c, \quad |ax+b|<c, \quad |ax+b|\geq c, \quad \text{or} \quad |ax+b|\leq c.$$

(In all of these, assume that $a \neq 0$.)

There are many approaches to this; you should choose one and stick to it, to avoid confusing yourself.

Main Case. Suppose that c is **positive**. Examples:

$$|x-3|>5, \quad |2x-1|<10, \quad |1+5x|\geq 7, \quad \text{or} \quad |x+13|\leq 6.$$

1. Solve the corresponding **equation** $|ax+b|=c$, and mark the two points you get on a number line.

For instance, if solving the equation gave the numbers -2 and 6, I would draw this picture:

2. If the absolute value expression is on the **greater-than** ("open") side of the "$>$", **shade the two outside intervals**.

If I started with $|x-2|>4$, I'd get this:

If the absolute value expression is on the **less-than** ("pointy") side of the "$>$", **shade the inside interval**.

If I started with $|x-2|<4$, I'd get this:

3. Mark the endpoints with "[" and "]" if the inequality had "$\geq$" or "$\leq$".

For example, with $|x-2|\geq 4$, I'd get this:

Mark the endpoints with "(" and ")" if the inequality had "$>$" or "$<$".
For example, with $|x-2|<4$, I'd get this:

4. Finally, write the inequalities or intervals for the shaded parts.

For example, with $|x-2|\geq 4$, I'd get this:

In interval notation, this would be $(-\infty, -2] \cup [6, \infty)$.

For example, with $|x - 2| < 4$, I'd get this:

In interval notation, this would be $(-2, 6)$.

Special Cases. If $c = 0$ or $c < 0$, the answer may be **a single point**, **all real numbers**, **all real numbers except for a single point**, or there may be **no solutions**. Some of these are special cases of the situations above.

I'll write down the possibilities for reference, but there are too many cases for you to memorize them all, so it's better to handle these individually. I'll give brief reasons for some of them.

1. $|ax + b| > 0$: The solution is all real numbers, except for $x = -\dfrac{b}{a}$. Thus,

$$x < -\frac{b}{a} \quad \text{or} \quad x > -\frac{b}{a}.$$

In interval form, $\left(-\infty, -\dfrac{b}{a}\right) \cup \left(-\dfrac{b}{a}, \infty\right)$.

2. $|ax + b| \geq 0$: The solution is all real numbers. You could write this as $\mathbb{R}$ (the symbol for the real numbers), or $(-\infty, \infty)$ in interval form.

This makes sense, because $|ax+b|$ is *always* greater than or equal to 0, being an absolute value.

3. $|ax + b| >$ (a negative number): : The solution is all real numbers. You could write this as $\mathbb{R}$ (the symbol for the real numbers), or $(-\infty, \infty)$ in interval form.

Likewise, since $|ax + b|$ is *always* greater than or equal to 0, it will always be greater than any negative number.

4. $|ax + b| \geq$ (a negative number): : The solution is all real numbers. You could write this as $\mathbb{R}$ (the symbol for the real numbers), or $(-\infty, \infty)$ in interval form.

The reasoning is the same as in Case 3.

5. $|ax + b| < 0$: There are no solutions.

Since $|ax + b|$ is *always* greater than or equal to 0, it can't be less than 0, since then it would be negative.

6. $|ax + b| \leq 0$: The solution is $x = -\dfrac{b}{a}$.

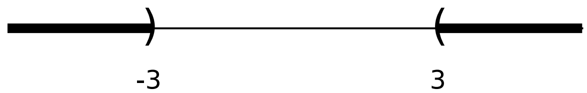
7. $|ax + b| <$ (a negative number): There are no solutions.

8. $|ax + b| \leq$ (a negative number): There are no solutions.

The reasoning in Cases 7 and 8 is similar to the reasoning in Case 5.

Example. Solve $|x| > 3$.

First, solve the helper equation $|x| = 3$. I get $x = \pm 3$.
I put -3 and 3 on the number line. Since the absolute value expression is on the **greater-than** ("open") side of the "$>$", I shade the two outside intervals:



The solution is $x < -3$ or $x > 3$.
In interval notation, this is $(-\infty, -3) \cup (3, \infty)$. $\square$

Example. Solve $|x - 2| < 10$.

First, solve the helper equation $|x - 2| = 10$.

$$\begin{array}{ccc} & |x-2| = 10 & \\ & x - 2 = \pm 10 & \\ \swarrow & & \searrow \\ x - 2 = 10 & & x - 2 = -10 \\ x = 12 & & x = -8 \end{array}$$

I put -8 and 12 on the number line. Since the absolute value expression is on the **less-than** ("pointy") side of the "$<$", I shade the inside interval:

The solution is $-12 < x < 8$.
In interval notation, this is $(-12, 8)$. $\square$

Example. Solve $|x + 3| \le 7$.

First, solve the helper equation $|x + 3| \le 7$.

$$\begin{array}{ccc} & |x+3| = 7 & \\ & x + 3 = \pm 7 & \\ \swarrow & & \searrow \\ x + 3 = 7 & & x + 3 = -7 \\ x = 4 & & x = -10 \end{array}$$

I put -10 and 4 on the number line. Since the absolute value expression is on the **less-than** ("pointy") side of the "$\le$", I shade the inside interval:

I use square brackets, because of the "equals" in "$\le$".
The solution is $-10 \le x \le 4$.
In interval notation, this is $[-10, 4]$. $\square$

Example. Solve $|x - 17| < -5$.

The absolute value of *anything* is greater than or equal to 0.

In particular, the absolute value of $|x - 17|$ must be greater than or equal to 0, so it can't be less than -5 — that would make it negative!

Hence, there are no solutions. $\square$

Example. Solve $|3x - 12| > 21$.

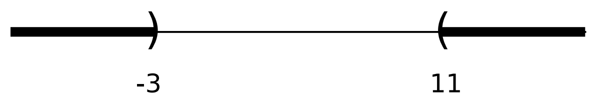
First, solve the helper equation $|3x - 12| = 21$.

$$|3x - 12| = 21$$
$$3x - 12 = \pm 21$$

$$\swarrow \qquad \searrow$$

$$\begin{array}{ccc} 3x - 12 = 21 & \qquad & 3x - 12 = -21 \\ 3x = 33 & & 3x = -9 \\ x = 11 & & x = -3 \end{array}$$

I put -3 and 11 on the number line. Since the absolute value expression is on the **greater-than** ("open") side of the "$>$", I shade the outside intervals:



The solution is $x < -3$ or $x > 11$.

In interval notation, this is $(-\infty, -3) \cup (11, \infty)$. $\square$
