3-30-2008

Complex Numbers

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$ (so $i^2 = -1$). For example, here are some complex numbers:

$$2 + 3i, \quad -77.5i, \quad 13\sqrt{7}, \quad \sqrt{-54}, \quad \frac{1+i}{2}.$$

Notice that real numbers are special kinds of complex numbers — namely, those that don't have an i-term.

Complex numbers are often called **imaginary numbers**, though there is nothing "imaginary" about them. It's unfortunate terminology, but it's very common.

For instance, in a complex number $a + bi$, a is called the **real part** and b is called the **imaginary part**. Thus, in $4 + 7i$, the real part is 4 and the imaginary part is 7.

Example. Express $\sqrt{-36}$ in the form $a + bi$, where a and b are real numbers.

$$\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i. \quad \square$$

Example. Express $\sqrt{-125}$ in the form $a + bi$, where a and b are real numbers.

$$\sqrt{-125} = \sqrt{25}\sqrt{5}\sqrt{-1} = 5i\sqrt{5}. \quad \square$$

Example. Express $\sqrt{-72}$ in the form $a + bi$, where a and b are real numbers.

$$\sqrt{-72} = \sqrt{36}\sqrt{2}\sqrt{-1} = 6i\sqrt{2}. \quad \square$$

Example. (a) Express i^{37} in the form $a + bi$, where a and b are real numbers.

Since $i^2 = -1$, I'm going to find the largest even number less than 37 and use that as the basis for breaking up the power.

$$i^{37} = i^{36} \cdot i = (i^2)^{18} \cdot i = (-1)^{18} \cdot i = 1 \cdot i = i. \quad \square$$

(b) Express $(-2i)^7$ in the form $a + bi$, where a and b are real numbers.

$$(-2i)^7 = (-2)^7 \cdot i^7 = -128 \cdot i^6 \cdot i = -128 \cdot (i^2)^3 \cdot i = -128 \cdot (-1)^3 \cdot i = -128 \cdot (-1) \cdot i = 128i. \quad \square$$

(c) Express i^{34} in the form $a + bi$, where a and b are real numbers.

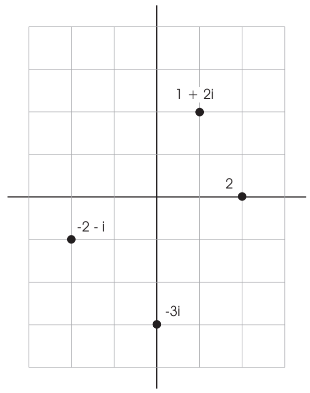
$$i^{34} = (i^2)^{17} = (-1)^{17} = -1. \quad \square$$

Example. You can add or subtract complex numbers by adding or subtracting the real parts and the imaginary parts:

$$(2 + 17i) + (5 + 6i) = (2 + 5) + (17i + 6i) = 7 + 23i.$$

$$(3-7i)+6(4+3i)=(3-7i)+(24+18i)=(3+24)+(-7i+18i)=27+11i. \quad \square$$

Example. Complex numbers can be represented by points in the plane. Use the real part for the x-direction and the imaginary part for the y-direction. Here are some examples:



$\square$

Example. Multiply complex numbers by using the distributive law:

$$3i(2+4i)=6i+12i^2=6i+12(-1)=-12+6i.$$

$$(2-i)^2=4-4i+i^2=4-4i-1=3-4i.$$

$$(5+6i)(5-6i)=25-36i^2=25-36(-1)=61.$$

$$(3+2i)(-1+6i)=(3)(-1)+(3)(6i)+(2i)(-1)+(2i)(6i)=-3+18i-2i+12i^2=-3+18i-2i-12=-15+16i.$$

Standard rules for exponents apply. For example,

$$i^{16}=(i^2)^8=(-1)^8=1,$$

$$i^{55}=i^{54}i=(i^2)^{27}i=(-1)^{27}i=-i.$$

I used the fact that -1 raised to an even power is 1 and -1 raised to an odd power is -1. $\square$

To divide one complex number by another, or to compute reciprocals of complex numbers, use the technique of **multiplying the top and bottom by the conjugate**.

What's the **conjugate** of a complex number? The conjugate of $2+3i$ is $2-3i$; the conjugate of $-6-7i$ is $-6+7i$. In others words, find the conjugate by flipping the sign of the imaginary part.

Example. Express $\dfrac{1}{3+4i}$ in the form $a+bi$.

Multiply the top and bottom by $3-4i$:

$$\frac{1}{3+4i}=\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i}=\frac{3-4i}{25}. \quad \square$$

Example. Express $\dfrac{2-i}{3-2i}$ in the form $a+bi$.

Multiply the top and bottom by $3+2i$:

$$\frac{2-i}{3-2i} = \frac{2-i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{(2-i)(3+2i)}{13} = \frac{8+i}{13}. \quad \square$$