The Exponential Function

If \( a \) is a positive number and \( a \neq 1 \), the **exponential function** with base \( a \) is

\[
y = a^x.
\]

You know what this means when \( x \) is an integer; for example,

\[
a^3 = a \cdot a \cdot a, \quad \text{and} \quad a^{-4} = \frac{1}{a^4}.
\]

You also know what this mean if \( x \) is a rational number; for instance,

\[
a^{2/3} = \sqrt[3]{a^2}.
\]

What does \( a^x \) mean if \( x \) is not a rational number? That is, suppose \( x \) has a decimal expansion which is infinite and does not repeat itself, such as

\[
\pi = 3.14159265358979323846264 \ldots
\]

What would \( 2^\pi \) mean?

There are ways of defining this precisely, but I’ll take an intuitive approach which relies on limits. Look at

\[
2^3, \quad 2^{3.1}, \quad 2^{3.14}, \ldots
\]

These numbers are

\[
2^3 = 8, \quad 2^{3.1} \approx 8.57419, \quad 2^{3.14} \approx 8.81524.
\]

If you keep going in this way, the numbers will approach a limit, and that limit is \( 2^\pi \):

\[
2^\pi = 8.82498 \ldots
\]

In a similar way, you can think of \( a^x \) as a limit of numbers which you get by using more and more of the decimal expansion of \( x \).

There is a special number which is often used as a base for an exponential function:

\[
e = 2.718281828459045 \ldots
\]

It can be defined by

\[
e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x.
\]

\( e \) is an irrational number, like \( \pi \); its decimal expansion is infinite, and does not repeat. It may seem puzzling why anyone would want to use such a weird base for an exponential function, rather than an apparently nicer base such as 2 or 10. It turns out that calculus explains what is special about \( e \); in fact, the exponential function \( e^x \) satisfies

\[
\frac{d}{dx} e^x = e^x.
\]

No other exponential — \( 2^x, 10^x \) — works out to be exactly the same as its derivative. This is why when mathematicians and scientists refer to \( e^x \) as the exponential function, and why it is used more than other exponential functions in math and science.

**Properties of exponentials.** Let \( a \) be a positive number, \( a \neq 1 \).

(a) \( a^x \) is defined for all \( x \), \( a^x > 0 \) for all \( x \).
(b) $a^x$ increases for all $x$ if $a > 1$ and decreases for all $x$ if $0 < x < 1$.

\[ y = 3^x \quad y = 0.5^x \]

(c) $a^0 = 1$.

(d) $a^p a^q = a^{p+q}$.

(e) $\frac{a^p}{a^q} = a^{p-q}$.

(f) $(a^p)^q = a^{pq}$.

Example. Let $a$ be a positive number, $a \neq 1$. Simplify $\frac{a^3a^{-7}}{a^2a^{-9}}$.

\[
\frac{a^3a^{-7}}{a^2a^{-9}} = a^{3-7}a^{-2+9} = a^{-4} = a^{-4-(-7)} = a^3. \quad \Box
\]

If an amount $P$ (the principal) is invested at an annual interest rate of $r\%$ compounded $k$ times a year, then after $n$ years the investment is worth

\[ A = P \left(1 + \frac{r}{k}\right)^{nk}. \]

Example. $1000$ is invested at 6\% annual interest, compounded monthly. How much is the investment worth after 10 years?

\[ A = 1000 \left(1 + \frac{0.06}{12}\right)^{10 \cdot 12} \approx 1819.40 \text{ dollars.} \quad \Box \]

Example. How much money must be invested at 6\% annual interest, compounded monthly, so that the investment is worth $2000$ after 4 years?

\[ 2000 = P \left(1 + \frac{0.06}{12}\right)^{4 \cdot 12}, \quad 2000 \approx 1.27049P, \quad P \approx 1574.20 \text{ dollars.} \]

$1574.20$ is said to be the present value of $2000$. \quad \Box
Now suppose there is an imaginary investment where the interest is compounded \textit{continuously}. You might think that you’d make a lot of money on such an investment!

The interest formula is

$$A = P \left(1 + \frac{r}{k}\right)^{nk}.$$ 

Write this as

$$A = P \left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{rn(k/r)}{k/r}} = P \left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}rn}.$$ 

Letting $k \to \infty$ corresponds to continuous compounding, since $k$ is the number of times you compound each year. But as $k \to \infty$, $\frac{k}{r} \to \infty$, so

$$\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}} \to e.$$ 

Thus, if a principal $P$ is invested at $r\%$ annual interest compounded continuously for $n$ years, the value of the investment is

$$A = Pe^{rn}.$$ 

\textbf{Example.} \$500 is invested for 3 years at 6\% interest compounded continuously. Find the value of the investment.

$$A = 500e^{3 \cdot 0.06} \approx 598.61 \text{ dollars}.$$