4-1-2014

Factoring Polynomials

The opposite of multiplying polynomials is **factoring**. Why would you want to factor a polynomial?

Let $p(x)$ be a polynomial. $p(c) = 0$ is equivalent to $x - c$ dividing $p(x)$.

Recall that when $p(c) = 0$, you say that c is a **root** of $p(x)$. The result above means that factoring is related to root-finding, and vice versa.

Example. If $x = 2$,

$$x^2 - x - 2 = 4 - 2 - 2 = 0.$$

It follows that $x - 2$ divides $x^2 - x - 2$; in fact,

$$x^2 - x - 2 = (x - 2)(x + 1). \quad \square$$

Besides root-finding, you may also want to factor an expression in order to simplify something.

Example. Provided that $x \neq 3$, I can simplify $\dfrac{x^3 - 3x^2}{x - 3}$ by cancellation:

$$\frac{x^3 - 3x^2}{x - 3} = \frac{x^2(x - 3)}{x - 3} = x^2.$$

I got the last equality by cancelling the $x - 3$'s, which is valid provided that $x \neq 3$. I "found" the $x - 3$ on the top by factoring. $\square$

I'll look at various methods for factoring polynomials: Removing a common factor, factoring quadratics by trial and error, using special forms for quadratics, using special forms for cubics, and factoring by grouping.

1. Removing a common factor.

In some cases, the terms of a sum have some factors in common. The common factors may be factored out of all the terms:

$$ab + ac = a(b + c).$$

This is the opposite of the distributive law.

Example.

$$4x^4 + 16x^3 + 48x^2 = 4x^2(x^2 + 4x + 12).$$

$$2x^3y + 6x^2y^2 + 14xy^3 = 2xy(x^2 + 3xy + 7y^2).$$

$$7x(x + y)^2 - 15x^2(x + y) = x(x + y)\left(7(x + y) - 15x\right) = x(x + y)\left(7x + 7y - 15x\right) = x(x + y)(7y - 8x). \quad \square$$

2. Factoring quadratics.

Now I'll discuss the important case of **factoring quadratics**. The discussion will be incomplete until later, when I discuss the **general quadratic formula**.

Example. How do you factor $x^2 + 7x + 12$?

To do this, think of two numbers which multiply to 12 and add to give 7. 3 and 4 work: $3 \cdot 4 = 12$ and $3 + 4 = 7$. So

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

How do you factor $x^2 - 5x + 6$?

Think of two numbers which multiply to 6 and add to give -5. I must have two negative numbers to make this work; -2 and -3 fit the bill. So

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

How do you factor $x^2 - 5x - 6$?

Think of two numbers which multiply to -6 and add to give -5. I must have two negative numbers to make this work; -6 and 1 fit the bill. So

$$x^2 - 5x - 6 = (x - 6)(x + 1).$$

How do you factor $x^2 - 8x + 16$?

Think of two numbers which multiply to 16 and add to give -8. I must have two negative numbers to make this work; -4 and -4 fit the bill. So

$$x^2 - 8x + 16 = (x - 4)(x - 4) = (x - 4)^2. \quad \square$$

I have factoring formulas which correspond to my special forms for multiplication.

1. $a^2 - b^2 = (a - b)(a + b)$
2. $a^2 - 2ab + b^2 = (a - b)^2$
3. $a^2 + 2ab + b^2 = (a + b)^2$

Example. This is the $a^2 - b^2$ form with $a = x$ and $b = 8$:

$$x^2 - 64 = (x - 8)(x + 8).$$

This is the $a^2 - 2ab + b^2$ form with $a = x$ and $b = 5$:

$$x^2 - 10x + 25 = (x - 5)^2.$$

This is the $a^2 + 2ab + b^2$ form with $a = x$ and $b = 10$:

$$x^2 + 20x + 100 = (x + 10)^2.$$

In the next example, I'll use the $a^2 - b^2$ form with $a = 2x$ and $b = 5$:

$$4x^2 - 25 = (2x - 5)(2x + 5).$$

Here's the $a^2 - b^2$ form with $a = 3x$ and $b = 4y$:

$$9x^2 - 16y^2 = (3x - 4y)(3x + 4y).$$

This example uses the $a^2 + 2ab + b^2$ form with $a = 2x$ and $b = y$:

$$4x^2 + 4xy + y^2 = (2x + y)^2.$$

And this example uses the $a^2 - 2ab + b^2$ form with $a = 5x$ and $b = 2y$:

$$25x^2 - 20xy + 4y^2 = (5x - 2y)^2. \quad \square$$

Note, however, that $a^2 + b^2$ does *not* factor (unless you use complex numbers).

Example. How do you factor $2x^2 + 9x - 5$?

I expect a factorization that looks like this:

$$2x^2 + 9x - 5 = (2x \pm *)(x \pm *).$$

What are the missing terms? I write down all possible ways of factoring -5, assuming that this will come out in terms of integers:

$(2x$		$)(x$		$)$
	5		-1	
	-5		1	
	1		-5	
	-1		5	

I find that

$$2x^2 + 9x - 5 = (2x - 1)(x + 5).$$

You can check this by multiplying out the right side. $\square$

Example. Factor $12x^2 + 11x - 5$.

I need numbers a and b whose product is 12, and such that

$$12x^2 + 11x - 5 = (ax + 5)(bx - 1) \quad \text{or} \quad 12x^2 + 11x - 5 = (ax - 5)(bx + 1).$$

If you try various combinations, you'll find that

$$12x^2 + 11x - 5 = (4x + 5)(3x - 1). \quad \square$$

Examples. Factoring quadratics is something you should be able to do fluently. Here are some more for practice.

To factor $x^2 + 9x + 14$, I need two numbers which add to 9 and multiply to 14. The pairs of numbers multiplying to 14 are $(1, 14)$ and $(2, 7)$. $2 + 7 = 9$, so I use 2 and 7:

$$x^2 + 9x + 14 = (x + 2)(x + 7).$$

To factor $x^2 - 9x + 18$, I need two numbers which add to -9 and multiply to 18. The pairs of numbers which multiply to 18 are $(1, 18)$, $(-1, -18)$, $(2, 9)$, $(-2, -9)$, $(3, 6)$, and $(-3, -6)$, $(-3) + (-6) = -9$, so I use -3 and -6:

$$x^2 - 9x + 18 = (x - 3)(x - 6).$$

To factor $x^2 - 14x + 49$, you can look for two numbers which add to -14 and multiply to 49. -7 and -7 work.

I would do it differently: The perfect square 49 makes me think that maybe this is a standard form. $49 = 7^2$, and $2 \cdot 7 = 14$, which is the middle coefficient — and that led me to $(x - 7)^2$. I used -7 because the middle coefficient was -14. So either way,

$$x^2 - 14x + 49 = (x - 7)^2.$$

To factor $x^2 - 2x - 24$, I need two numbers which add to -2 and multiply to -24. The pairs of numbers which multiply to -24 are $(1, -24)$, $(-1, 24)$, $(2, -12)$, $(-2, 12)$, $(3, -8)$, $(-3, 8)$, $(4, -6)$, and $(-4, 6)$. $4 + (-6) = -2$, so

$$x^2 - 2x - 24 = (x - 6)(x + 4).$$

Don't forget that you can always check your factoring by multiplication!

To factor $4x^2 + 4x + 1$, I notice that both the 4 in $4x^2$ and the 1 are perfect squares. $4x^2 = (2x)^2$ and $1 = 1^2$; is this a standard form? Well, $2 \cdot 2x = 4x$, so

$$4x^2 + 4x + 1 = (2x + 1)^2.$$

To factor $2x^2 - 5x + 3$, I notice that $2x^2$ breaks down as $2x \cdot x$ and 3 could be either $1 \cdot 3$ or $(-1) \cdot (-3)$. To get the -5 in the middle, I must have -1 and -3. So there are two possibilities:

$$(2x - 1)(x - 3) \qquad \text{or} \qquad (2x - 3)(x - 1).$$

In fact,

$$2x^2 - 5x + 3 = (2x - 3)(x - 1).$$

$4x^2 + 4x - 3$ is more complicated. The $4x^2$ could be $4x \cdot x$ or $2x \cdot 2x$. The -3 is either $1 \cdot (-3)$ or $(-1) \cdot 3$. Here are the possibilities:

$$(4x + 1)(x - 3) = 4x^2 - 11x - 3, \quad (4x - 3)(x + 1) = 4x^2 + x - 3,$$

$$(4x - 1)(x + 3) = 4x^2 + 11x - 3, \quad (4x + 3)(x - 1) = 4x^2 - x - 3,$$

$$(2x + 1)(2x - 3) = 4x^2 - 4x - 3, \quad (2x - 1)(2x + 3) = 4x^2 + 4x - 3.$$

So

$$4x^2 + 4x - 3 = (2x - 1)(2x + 3). \quad \square$$

Example. Factor $16x^4 - 81$.

First,

$$16x^4 - 81 = (4x^2)^2 - 9^2 = (4x^2 - 9)(4x^2 + 9) = (2x - 3)(2x + 3)(4x^2 + 9).$$

Fact: A sum of two squares does not factor. Thus, $4x^2 + 9 = (2x)^2 + 3^2$ can't be factored. The factorization is $(2x - 3)(2x + 3)(4x^2 + 9)$. $\square$

Example. (**More than one variable**) Factor $a^3 - 6a^2b - 7ab^2$.

I take out a common factor, then factor the remaining quadratic term by trial:

$$a^3 - 6a^2b - 7ab^2 = a(a^2 - 6ab - 7b^2) = a(a - 7b)(a + b).$$

Notice that $a^2 - 6ab - 7b^2 = (a - 7b)(a + b)$ is like $a^2 - 6 - 7 = (a - 7)(a + 1)$, except with the additional b's. □

---

3. Cubic formulas.

1. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
2. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example.

$$x^3 - 64 = (x - 4)(x^2 + 4x + 16).$$

$$x^3 + 8y^3 = (x + 2y)(x^2 - 2xy + 4y^2).$$

$$\frac{1}{x^3} - \frac{1}{125} = \left(\frac{1}{x} - \frac{1}{5}\right)\left(\frac{1}{x^2} + \frac{1}{5x} + \frac{1}{25}\right). \quad □$$

---

4. Factoring by grouping.

In some cases, you can factor an expression *by factoring pieces of the expression separately, then looking for common factors in the pieces.* This is easier to show than to explain, so here are some examples.

Example. Factor $x^3 - 4x^2 + 5x - 20$.

I don't have a rule for factoring a cubic of this form. I'll break the polynomial up into two pieces:

$$x^3 - 4x^2 + 5x - 20 = (x^3 - 4x^2) + (5x - 20).$$

Now I'll take a common factor out of each piece, then look for a common factor of the whole expression.

$$x^3 - 4x^2 + 5x - 20 = (x^3 - 4x^2) + (5x - 20) = x^2(x - 4) + 5(x - 4) = (x^2 + 5)(x - 4). \quad □$$

Example. Factor $x^3 - 7x^2 - 9x + 63$.

$$x^3 - 7x^2 - 9x + 63 = (x^3 - 7x^2) - (9x - 63) = x^2(x - 7) - 9(x - 7) = (x^2 - 9)(x - 7) = (x - 3)(x + 3)(x - 7). \quad □$$

Example. Factor $x^2 - 3xy + 5x - 15y$.

$$x^2 - 3xy + 5x - 15y = (x^2 - 3xy) + (5x - 15y) = x(x - 3y) + 5(x - 3y) = (x + 5)(x - 3y). \quad □$$
