2-25-2008

Word Problems Involving Fractions

Despite the name, not all of these word problems **must** be solved using fractions. They **can** be, but where possible I've given solutions which avoid fractions. This keeps things simple.

The first group involve fractions directly. The second group involves pumping water in and out of tanks. There are several problems involving speed, time, and distance. Finally, there are work problems — problems involving the completion of some task.

Example. One number is twice another number. The sum of their reciprocals is $\dfrac{1}{2}$. Find the numbers.

Call the numbers x and y. The first sentence says

$$x = 2y.$$

The second sentence says

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$$

Substitute for x:

$$\frac{1}{2y} + \frac{1}{y} = \frac{1}{2}.$$

Add the fractions on the left:

$$\frac{1}{2y} + \frac{2}{2y} = \frac{1}{2}, \quad \frac{3}{2y} = \frac{1}{2}.$$

Multiply both sides by $2y$ to get $3 = y$. Therefore, $x = 2y = 6$. Check:

$$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}. \quad \square$$

Example. If a number is added to the top and the bottom of $\dfrac{5}{7}$, you get $\dfrac{6}{7}$. What is the number?

Let n be the number to be added.

If a number is added to the top and the bottom of $\dfrac{5}{7}$, you get $\dfrac{6}{7}$: $\dfrac{5+n}{7+n} = \dfrac{6}{7}$.

Clear the fractions and solve:

$$\begin{aligned} 7(7+n) \cdot \frac{5+n}{7+n} &= 7(7+n) \cdot \frac{6}{7} \\ 7(5+n) &= 6(7+n) \\ 35 + 7n &= 42 + 6n \\ n &= 7 \end{aligned}$$

Check: When $n = 7$,

$$\frac{5+n}{7+n} = \frac{12}{14} = \frac{6}{7}.$$

The number is 7. $\square$

Example. The numerator of a fraction (not necessarily in lowest terms) is 7 smaller than its denominator. If the fraction is subtracted from 1, you get $\frac{1}{3}$. What is the fraction?

Let $\frac{n}{d}$ be the fraction. The first sentence says that $n = d - 7$. The second sentence says

$$1 - \frac{n}{d} = \frac{1}{3}.$$

Substitute for n:

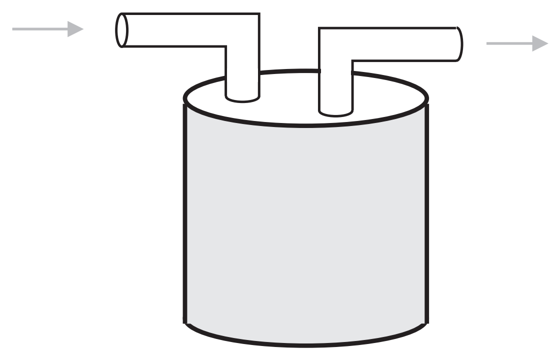
$$1 - \frac{d-7}{d} = \frac{1}{3}.$$

Multiply both sides by $3d$:

$$3d - 3(d-7) = d, \quad 3d - 3d + 21 = d, \quad d = 21.$$

Therefore, $n = d - 7 = 14$, and the fraction is $\frac{14}{21}$. □

The next few problems discuss situations where several pipes pump water into or out of a tank.



In this situation, *the rates at which water flows through the pipes adds or subtracts, according to whether the water is flowing in or out.*

Example. If the inlet pipe for a tank is opened, the tank will be filled in 6 hours.
If the outlet pipe for a tank is opened when the tank is full, the tank will be emptied in 8 hours.
How long will it take to fill the whole tank if both the inlet and outlet pipes are open?

Let x be the rate at which the inlet pipe fills the tank. Let y be the rate at which the outlet pipe drains the tank. Let t be the time it takes to fill the tank when both pipes are open.

	hours	$\cdot$	tanks per hour	=	tanks
inlet	6	$\cdot$	x	=	1
outlet	8	$\cdot$	y	=	1
both	t	$\cdot$	$x - y$	=	1

The inlet row of the table says $6x = 1$, so $x = \frac{1}{6}$.

The outlet row of the table says $8y = 1$, so $y = \frac{1}{8}$.

The last row of the table says $(x-y)t = 1$. Plug in $x = \frac{1}{6}$ and $y = \frac{1}{8}$:

$$\left(\frac{1}{6} - \frac{1}{8}\right) t = 1, \quad \text{or} \quad \frac{1}{24} t = 1.$$

Therefore, $t = 24$. It takes 24 hours to fill the tank. $\square$

Example. If the inlet pipe for a tank is opened, the tank will be filled in 1.5 hours.

If the outlet pipe for a tank is opened when the tank is full, the tank will be emptied in 4 hours.

If the tank starts out half full, how long will it take to fill the whole tank if both the inlet and outlet pipes are open?

Let x be the rate at which the inlet pipe fills the tank. Let y be the rate at which the outlet pipe drains the tank. Let t be the time it takes to fill the tank when both pipes are open.

	hours	$\cdot$	tanks per hour	$=$	tanks
inlet	1.4	$\cdot$	x	$=$	1
outlet	4	$\cdot$	y	$=$	1
both	t	$\cdot$	$x - y$	$=$	0.5

The inlet row of the table says $1.5x = 1$, so $x = \frac{2}{3}$.

The outlet row of the table says $4y = 1$, so $y = \frac{1}{4}$.

The last row of the table says $(x-y)t = 0.5$. Plug in $x = \frac{2}{3}$ and $y = \frac{1}{4}$:

$$\left(\frac{2}{3} - \frac{1}{4}\right) t = 0.5, \quad \text{or} \quad \frac{1}{2} t = 0.5.$$

Therefore, $t = 1$. It takes 1 hour to fill the tank. $\square$

Example. Pipe A can fill 2 tanks in 5 minutes. Pipe B can fill 4 tanks in 8 minutes. How long will it take them to fill a single tank, if they work together?

Let x be the rate at which A fills tanks and let y be the rate at which B fills tanks. Finally, let t be the time required for them to fill a single tank, if they work together.

	minutes	$\cdot$	tanks per minutes	$=$	tanks
A	5	$\cdot$	x	$=$	2
B	8	$\cdot$	y	$=$	4
both	t	$\cdot$	$x + y$	$=$	1

The first line gives $5x = 2$, so $x = \frac{2}{5}$.

The second line gives $8y = 4$, so $y = \frac{1}{2}$.

The third line gives $(x + y) \cdot t = 1$. Substitute for x and y and simplify:

$$\left(\frac{2}{5} + \frac{1}{2}\right) t = 1, \quad \frac{9}{10} t = 1, \quad t = \frac{10}{9}. \quad \square$$

Example. Pipe A can fill an empty tank by itself in 4 hours.

Pipe A is opened, then closed after 3 hours; at that point, pipe B is opened, and pipe B takes 6 hours to finish filling the tank.

How long will it take pipe B to fill an empty tank by itself?

Let x be pipe A's rate and let y be pipe B's rate.

	hours	$\cdot$	tanks per hour	$=$	tanks
pipe A alone	4	$\cdot$	x	$=$	1
pipe A first pipe B second	3 6	$\cdot$ $\cdot$	x y	$=$ $=$	$3x$ $6y$
pipe B alone	t	$\cdot$	y	$=$	1

The first row gives $4x = 1$, so $x = \dfrac{1}{4}$.

In the second row, pipe A is open for 3 hours and pipe B is open for 6 hours. Together, they fill the tank (i.e. 1 tank). So

$$3x + 6y = 1.$$

Plug in $x = \dfrac{1}{4}$ and solve for y:

$$\begin{aligned} \frac{3}{4} + 6y &= 1 \\ 6y &= \frac{1}{4} \\ y &= \frac{1}{24} \end{aligned}$$

The last equation says $yt = 1$. Plug in $y = \dfrac{1}{24}$ and solve for y:

$$\begin{aligned} \frac{1}{24}t &= 1 \\ t &= 24 \end{aligned}$$

Pipe B takes 24 hours to fill a tank by itself. □

It's easy to remember the basic equation for **motion problems** if you think of a familiar everyday situation.

If you drive at a speed of 60 miles per hour for 2 hours, how far will you travel? Everyone knows how to figure this out:

$$\left(60\ \frac{\text{miles}}{\text{hour}}\right) \cdot 2 \text{ hours} = 120 \text{ miles}.$$

The general relationship between speed, time, and distance is

$$\text{speed} \cdot \text{time} = \text{distance}.$$

If you have trouble remembering this equation, just think of an everyday travel situation like the one above.

Example. Calvin runs 3.5 hours at 2 miles per hour. How far does he travel?

$$\left(2\ \frac{\text{miles}}{\text{hour}}\right) \cdot 3.5 \text{ hours} = 7 \text{ miles.} \quad \square$$

Example. How long does it take Phoebe to walk 10 miles if she walks at a constant speed of 3 miles per hour?

Let t be the time she takes (in hours).

$$\left(3\ \frac{\text{miles}}{\text{hour}}\right) \cdot t \text{ hours} = 10 \text{ miles.}$$

Solving for t, I get $t = \dfrac{10}{3}$ hours — i.e. 3 hours and 20 minutes. $\square$

Example. If Bonzo drives 210 miles in 5 hours at a constant speed, how fast was he driving?

Let v be Bonzo's speed in miles per hour.

$$\left(v\ \frac{\text{miles}}{\text{hours}}\right) \cdot 5 \text{ hours} = 210 \text{ miles.}$$

Solving for v, I obtain $v = \dfrac{210}{5} = 42$ miles per hour. $\square$

Example. Calvin leaves Wimp City at noon and runs at a constant speed of 2 miles per hour. Phoebe leaves Wimp City at 1 p.m. and runs after him at a constant speed of 3 miles per hour. When does Phoebe catch up with Calvin?

Let t be the number of hours that Phoebe must run to catch up with Calvin. Since Calvin started 1 hour earlier, he will have run $t + 1$ hours at the time she catches up.

	time	$\cdot$	speed	$=$	distance
Calvin	$t+1$	$\cdot$	2	$=$	$2(t+1)$
Phoebe	t	$\cdot$	3	$=$	$3t$

Calvin has travelled $2(t + 1)$ miles. Phoebe has travelled $3t$ miles. Since she's caught up with him, the distances they've travelled are the same:

$$\begin{aligned} 2(t+1) &= 3t \\ 2t + 2 &= 3t \\ 2 &= t \end{aligned}$$

Since Phoebe must run for $t = 2$ hours and she started at 1 p.m., she catches up at 3 p.m. $\square$

Example. Bonzo and Calvin leave Phoebe's house at the same time in opposite directions. Bonzo runs twice as fast as Calvin; after 10 seconds, they are 90 feet apart. How fast are they each running?

Let x be Calvin's speed in feet per second. Since Bonzo runs twice as fast, his speed is $2x$.

	time	$\cdot$	speed	$=$	distance
Calvin	10	$\cdot$	x	$=$	$10x$
Bonzo	10	$\cdot$	$2x$	$=$	$20x$

Calvin has travelled $10x$ feet and Bonzo has travelled $20x$ feet. Since they run in opposite directions, they are $10x + 20x = 30x$ feet apart. Set this equal to 90 and solve for x:

$$\begin{aligned} 30x &= 90 \\ x &= 3 \end{aligned}$$

Calvin is running at 3 feet per second, so Bonzo is running at 6 feet per second. $\square$

Example. Calvin can row at 2 miles per hour in still water. He rows his boat up and down a river which flows at a constant speed. It takes him 4 hours longer to row 15 miles upstream than it takes him to row 15 miles downstream. How fast is the river flowing?

Let x be the speed of the river. Calvin's upstream speed is $2 - x$ and his downstream speed is $2 + x$.

Let t be the amount of time it takes him to row 15 miles upstream.

It takes him 4 hours longer to row 15 miles upstream than it takes him to row 15 miles downstream, so it takes him $t - 4$ hours to row 15 miles downstream.

	time	$\cdot$	speed	$=$	distance
upstream	t	$\cdot$	$2 - x$	$=$	15
downstream	$t - 4$	$\cdot$	$2 + x$	$=$	15

The upstream row of the table says $(2 - x)t = 15$, so $t = \dfrac{15}{2 - x}$.

The downstream row of the table says $(2 + x)(t - 4) = 15$. Plug in $t = \dfrac{15}{2 - x}$:

$$(2 + x)\left(\frac{15}{2 - x} - 4\right) = 15.$$

Clear denominators and solve for x:

$$\begin{aligned} (2 + x)\left(\frac{15}{2 - x} - 4\right) \cdot (2 - x) &= 15 \cdot (2 - x) \\ (2 + x)\left(\frac{15}{2 - x} \cdot (2 - x) - 4 \cdot (2 - x)\right) &= 15 \cdot (2 - x) \\ (2 + x)\left(15 - 4(2 - x)\right) &= 15(2 - x) \\ (2 + x)\left(15 - 8 + 4x\right) &= 30 - 15x \\ (2 + x)\left(7 + 4x\right) &= 30 - 15x \\ 4x^2 + 15x + 14 &= 30 - 15x \\ 4x^2 + 30x - 16 &= 0 \\ 2x^2 + 15x - 8 &= 0 \\ (2x - 1)(x + 8) &= 0 \end{aligned}$$

The possible solutions are $x = \dfrac{1}{2} = 0.5$ and $x = -8$.
x is the speed of the river, so it can't be negative. This rules out $x = -8$.
If $x = 0.5$,

$$(2 + x)\left(\frac{15}{2 - x} - 4\right) = (2.5)\left(\frac{15}{1.5} - 4\right) = (2.5)(10 - 4) = (2.5)(6) = 15.$$

The river is flowing at 0.5 miles per hour. $\square$

Work problems are basically the same as tank problems, except that instead of filling a tank, the tasks to be accomplished are more varied. In all cases, the basic equations follow the same pattern as in the tank problems and the time-speed-distance problems:

$$(\text{time}) \cdot (\text{rate}) = (\text{amount}).$$

Example. Phoebe can eat a broccoli and anchovy pizza in 30 minutes; Calvin can eat the same pizza in 20 minutes. How long will it take them if they work together?

Let x be Phoebe's rate, and let y be Calvin's rate.

	minutes	$\cdot$	pizzas per minute	$=$	pizzas
Phoebe	30	$\cdot$	x	$=$	1
Calvin	20	$\cdot$	y	$=$	1
together	t	$\cdot$	$x + y$	$=$	1

The first equation says $30x = 1$, so $x = \dfrac{1}{30}$.
The second equation say $20y = 1$, so $y = \dfrac{1}{20}$.
The third equation says

$$(x + y)t = 1.$$

Plug in $x = \dfrac{1}{30}$ and $y = \dfrac{1}{20}$ and solve for t:

$$\begin{aligned}
\left(\frac{1}{30} + \frac{1}{20}\right) t &= 1 \\
\left(\frac{2}{60} + \frac{3}{60}\right) t &= 1 \\
\frac{5}{60} t &= 1 \\
\frac{1}{12} t &= 1 \\
t &= 12
\end{aligned}$$

It will take them 12 minutes. $\square$

Example. Annihilus can destroy the world in 3 days; Dr. Doom can do it in 2. How long will it take them if they work together?

Let x be Annihilus's rate, and let y be Dr. Doom's rate.

	days	$\cdot$	worlds destroyed per day	$=$	worlds destroyed
Annihilus	3	$\cdot$	x	$=$	1
Dr. Doom	2	$\cdot$	y	$=$	1
together	t	$\cdot$	$x+y$	$=$	1

The first equation says $3x = 1$, so $x = \dfrac{1}{3}$.
The second equation say $2y = 1$, so $y = \dfrac{1}{2}$.
The third equation says

$$(x+y)t = 1.$$

Plug in $x = \dfrac{1}{3}$ and $y = \dfrac{1}{2}$ and solve for t:

$$\begin{aligned} \left(\frac{1}{3} + \frac{1}{2}\right) t &= 1 \\ \frac{5}{6} t &= 1 \\ t &= \frac{6}{5} \end{aligned}$$

It will take them $\dfrac{6}{5} = 1.2$ days. $\square$

Example. Calvin can eat 148 doughnuts in 4 hours. Bonzo and Calvin, eating together, can eat 488 doughnuts in 8 hours. How long will it take Bonzo to eat 168 doughtnuts by himself?

Let x be Calvin's rate and let y be Bonzo's rate.

	minutes	$\cdot$	doughnuts per minute	$=$	doughnuts
Calvin	4	$\cdot$	x	$=$	148
Calvin and Bonzo	8	$\cdot$	$x+y$	$=$	488
Bonzo	t	$\cdot$	y	$=$	168

The first equation says $4x = 148$, so $x = 37$.
The second equation says $8(x+y) = 488$. Plug in $x = 37$ and solve for y:

$$\begin{aligned} 8(37+y) &= 488 \\ 37 + y &= 61 \\ y &= 24 \end{aligned}$$

The last equation says $yt = 168$. Plug in $y = 24$ and solve for t:

$$\begin{aligned} 24t &= 168 \\ t &= 7 \end{aligned}$$

It takes Bonzo 7 hours. $\square$
