Fractional Exponents

Fractional exponents are related to roots or radicals.

If n is a positive integer, then

$$a^{1/n} \quad \text{is the } n^{\text{th}} \text{ root of } a.$$

If a is positive, it is the *positive* number b such that

$$b^n = a.$$

If a is negative, then:

1. If n is odd, $a^{1/n}$ is the *negative* number b such that

$$b^n = a.$$

2. If n is even, $a^{1/n}$ is *undefined*.

$a^{1/n}$ is also written $\sqrt[n]{a}$.

Example. Compute the exact values of:

(a) $9^{1/2}$.

(b) $125^{1/3}$.

(c) $-64^{1/3}$.

(d) $(-16)^{1/4}$.

(e) $-16^{1/4}$.

$$9^{1/2} = 3, \quad \text{since} \quad 3^2 = 9.$$

$9^{1/2}$ is the same as $\sqrt{9}$. Note that $\sqrt{9}$ is *not* "± 3".

$$125^{1/3} = 5, \quad \text{since} \quad 5^3 = 125.$$

$$-64^{1/3} = -4, \quad \text{since} \quad (-4)^3 = -64.$$

$$(-16)^{1/4} \quad \text{is undefined.}$$

$$-16^{1/4} = -(16^{1/4}) = -2.$$

In the last example, exponentiation takes precedence over negation. □

If m and n are positive integers,

$$a^{m/n} \quad \text{means} \quad \sqrt[n]{a^m}.$$

This makes sense, since

$$\sqrt[n]{a^m} = (a^m)^{1/n},$$

and this should equal $a^{m/n}$ if the rule for multiplying exponents is to hold in this case.

Equivalently,

$$a^{m/n} = (\sqrt[n]{a})^m.$$

In other words, you can do the root and the power in either order.
$\sqrt[n]{a^m}$ involves an n^{th} root, so it may be positive, negative, or undefined.

Example. Compute the exact values of:

(a) $8^{5/3}$.

(b) $64^{5/6}$.

(c) $(-27)^{2/3}$.

(d) $36^{-3/2}$.

(e) $(-125)^{-4/3}$.

$$8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32.$$

$$64^{5/6} = (\sqrt[6]{64})^5 = 2^5 = 32.$$

$$(-27)^{2/3} = (\sqrt[3]{-27})^2 = (-3)^2 = 9.$$

$$36^{-3/2} = \frac{1}{36^{3/2}} = \frac{1}{(\sqrt{36})^3} = \frac{1}{6^3} = \frac{1}{216}.$$

$$(-125)^{-4/3} = \frac{1}{(-125)^{4/3}} = \frac{1}{(\sqrt[3]{-125})^4} = \frac{1}{(-5)^4} = \frac{1}{625}. \quad \Box$$

Example. Use a calculator to approximate $5^{1/3}$ and $(-10)^{1/7}$.

$$5^{1/3} \approx 1.70998.$$

Roots of negative numbers can present a problem; some calculators will return a complex number, or give an error message. You can fix things by figuring out the *sign* of the result beforehand. Then make the base positive for your calculator and fix the sign at the end.

For example, $(-10)^{1/7}$ is an odd root of a negative number, so it's negative. Knowing this, I use the calculator to compute $10^{1/7}$:

$$10^{1/7} \approx 1.38950.$$

Therefore,

$$(-10)^{1/7} \approx -1.38950. \quad \Box$$

Example. Is "$(-1)^{2/6}$" undefined, since -1 is negative and the 6^{th} root is an even root?

Before considering a fractional exponent, the fraction should be reduced to lowest terms: $\dfrac{2}{6} = \dfrac{1}{3}$.

$$(-1)^{1/3} = -1, \quad \text{since} \quad (-1)^3 = -1. \quad \Box$$

The rules I gave earlier for working with integer exponents work with fractional exponents — with certain exceptions for *even roots*.

$$x^0 = 1$$
$$x^a x^b = x^{a+b}$$
$$\frac{x^a}{x^b} = x^{a-b}$$
$$(x^a)^b = x^{ab}$$
$$(xy)^a = x^a y^a$$
$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

Example. Simplify $(x^3y^{15})^{1/3}$.

$$(x^3y^{15})^{1/3} = (x^3)^{1/3}(y^{15})^{1/3} = x^1y^5 = xy^5. \quad \square$$

Example. Simplify $(x^{3/7}y^{9/11})^{1/3}$.

$$(x^{3/7}y^{9/11})^{1/3} = (x^{3/7})^{1/3}(y^{9/11})^{1/3} = x^{1/7}y^{3/11}. \quad \square$$

However,

$$(x^2)^{1/2} \quad \text{is not the same as} \quad x.$$

Why? $(x^2)^{1/2} = \sqrt{x^2}$, and $\sqrt{\text{junk}}$ is always nonnegative. But x could be negative: For example,

$$\sqrt{(-3)^2} = \sqrt{9} = 2, \quad \text{but} \quad x = -3.$$

In this case, $\sqrt{x^2} \neq x$.
In fact, if n is an even integer,

$$(x^n)^{1/n} = \sqrt[n]{x^n} = |x|.$$

(Of course, I can drop the absolute values if I know x is nonnegative.)

Example. Simplify $(x^4)^{1/4}$.

$$(x^4)^{1/4} = \sqrt[4]{x^4} = |x|.$$

Note that since I didn't assume x was nonnegative, the answer is not "x". $\square$

Example. Assuming that x and y are nonnegative, simplify $\sqrt{16x^4y^6}$.

$$\sqrt{16x^4y^6} = \sqrt{16}\sqrt{x^4}\sqrt{y^6} = 4x^2y^3. \quad \square$$

Example. Simplify $\sqrt[3]{8x^6y^3}$.

$$\sqrt[3]{8x^6y^3} = (8x^6y^3)^{1/3} = 8^{1/3}\cdot(x^6)^{1/3}\cdot(y^3)^{1/3} = 2x^2y. \quad \square$$

Example. Assuming that x and y are nonnegative, simplify $(-8x^6y^{3/5})^{1/3}(x^{-5}y^{5/3})^{1/5}$. Write your answer using positive powers.

$$(-8x^6y^{3/5})^{1/3}(x^{-5}y^{5/3})^{1/5} = (-8)^{1/3}(x^6)^{1/3}(y^{3/5})^{1/3}(x^{-5})^{1/5}(y^{5/3})^{1/5} =$$
$$(-2)x^2y^{1/5}x^{-1}y^{1/3} = -2xy^{8/15}. \quad \square$$

Example. Assuming that x and y are nonnegative, simplify $(4x^2y^{-3})^{3/2}\cdot 3x^{5/2}y^{1/3}$. Write your answer using positive powers.

$$(4x^2y^{-3})^{3/2}\cdot 3x^{5/2}y^{1/3} = 4^{3/2}(x^2)^{3/2}(y^{-3})^{3/2}\cdot 3x^{5/2}y^{1/3} = 8x^3y^{-9/2}\cdot 3x^{5/2}y^{1/3} =$$
$$24x^{11/2}y^{-25/6} = \frac{24x^{11/2}}{y^{25/6}}. \quad \square$$

Example. Assuming that x and y are nonnegative, simplify $\dfrac{(8x^6y^{3/2})^{1/3}}{6(x^3y^{21/2})^{1/6}}$. Write your answer using positive powers.

$$\frac{(8x^6y^{3/2})^{1/3}}{6(x^3y^{21/2})^{1/6}} = \frac{8^{1/3}(x^6)^{1/3}(y^{3/2})^{1/3}}{6(x^3)^{1/6}(y^{21/2})^{1/6}} = \frac{2x^2y^{1/2}}{6x^{1/2}y^{7/4}} =$$
$$\frac{x^{3/2}y^{-5/4}}{3} = \frac{x^{3/2}}{3y^{5/4}}. \quad \square$$

Example. Assuming that x and y are nonnegative, simplify $\dfrac{\sqrt{27x^{14}y^{-4}}}{(x^{-6}y^3)^{1/3}}$. Write your answer using positive powers.

$$\frac{\sqrt{27x^{14}y^{-4}}}{(x^{-6}y^3)^{1/3}} = \frac{\sqrt{27}\cdot(x^{14})^{1/2}\cdot(y^{-4})^{1/2}}{(x^{-6})^{1/3}\cdot(y^3)^{1/3}} = \frac{3\sqrt{3}\cdot x^7\cdot y^{-2}}{x^{-2}\cdot y} = 3\sqrt{3}x^9y^{-3} = \frac{3\sqrt{3}x^9}{y^3}. \quad \square$$

Example. Assuming that x, y, and z are nonnegative, simplify $\left(\dfrac{-32x^9y^{-3}}{z^{15}}\right)^{1/3}$. Write your answer using positive powers.

$$\left(\frac{-32x^9y^{-3}}{z^{15}}\right)^{1/3} = \frac{(-32)^{1/3}(x^9)^{1/3}(y^{-3})^{1/3}}{(z^{15})^{1/3}} = \frac{-2x^3y^{-1}}{z^5} = \frac{-2x^3}{yz^5}. \quad \square$$
