2-27-2008

Functions and Graphs

A **function** is a rule which assigns a unique output to each input.

Example. In mathematics, functions are often denoted using notation like the following:

$$f(x) = x^2.$$

This says that the name of the function is f, and x denotes a typical input. The right side tells how to produce an output from the input: Square it. Thus,

$$f(1) = 1^2 = 1, \quad f(5) = 5^2 = 25, \quad \text{but note that} f(-5) = (-5)^2 = 25 \quad \text{as well.}$$

Thus, different inputs can produce the same output.
Here is a **table** of some values of f:

x	0	2.1	-3	$\frac{1}{2}$	π
$f(x)$	0	4.41	9	$\frac{1}{4}$	π^2

Similarly,

$$f(x+1) = (x+1)^2, \qquad \text{and} \qquad f(a+b) = (a+b)^2.$$

Likewise, suppose $g(x) = \dfrac{1}{x+3}$. Then

$$g(2) = \frac{1}{5}, \quad g(-17) = -\frac{1}{14}, \quad \text{but} \quad g(-3) \quad \text{is undefined.}$$

Moreover,

$$g(x^2) = \frac{1}{x^2+3}, \qquad \text{and} \qquad g(x-1) = \frac{1}{(x-1)+3} = \frac{1}{x+2}. \quad \square$$

The **domain** of a function is the set of allowable inputs.

Example. Suppose $g(x) = \dfrac{1}{x+3}$. I can't plug in $x = -3$, as I saw above. Any other number may be plugged in for x. The domain consists of all x except $x = 3$ — sometimes I'll get lazy and write this as "$x \neq 3$" for short. $\square$

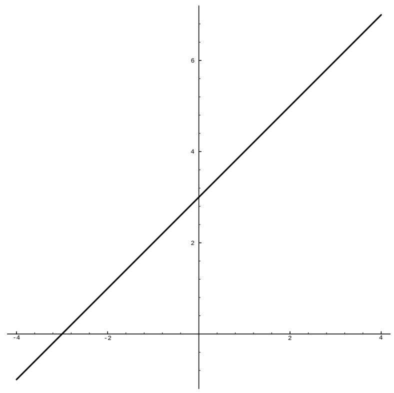
Example. Suppose $h(x) = \dfrac{x}{x^2 - x - 2}$. Then

$$h(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x-2)(x+1)}.$$

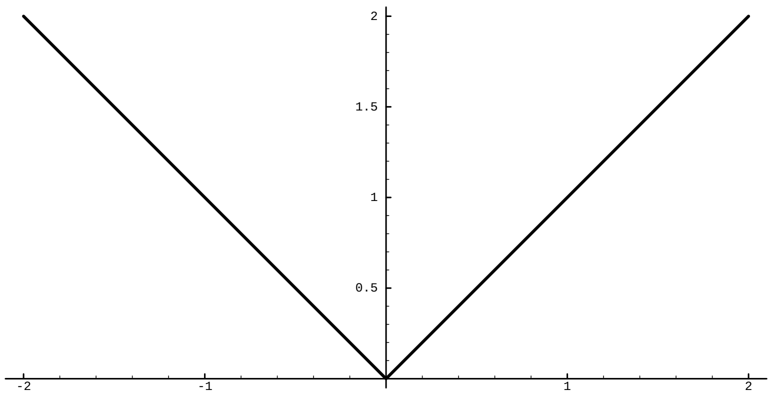
Plugging in $x = 2$ or $x = -1$ would cause division by 0; other values of x are okay. Hence, the domain consists of all real numbers except for 2 and -1. $\square$

You can **graph** a function by making a table of some function values, then plotting the points.

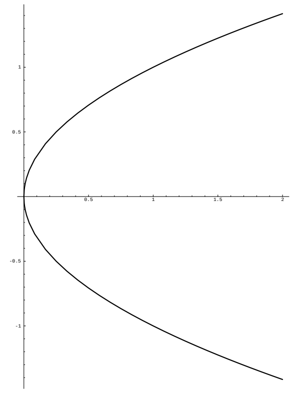
Example. Here's the graph of $f(x) = x + 3$:



Here's the graph of $y = |x|$:



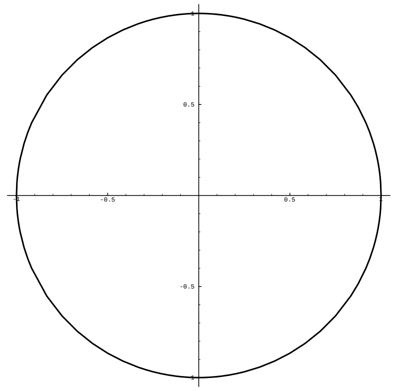
Here's the graph of $x = y^2$:



$x = y^2$ is not a function, since some inputs produce more than one output. For example, $x = 1$ gives $y = 1$ or $y = -1$.

A graph represents a function if and only if every vertical line hits the graph at most once.

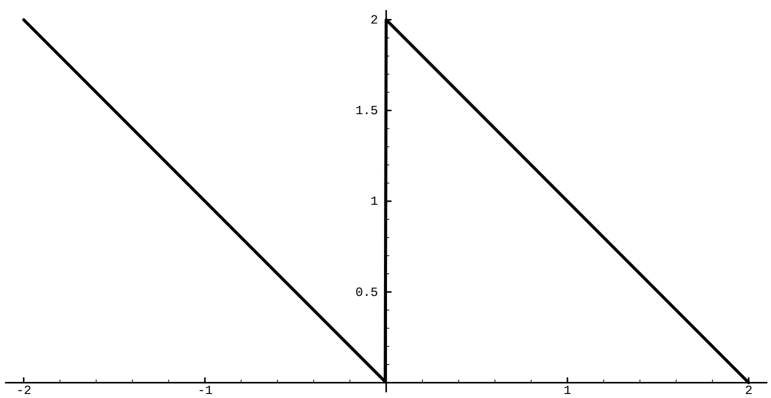
Here's the graph of $x^2 + y^2 = 1$:



This is also not the graph of a function.
Finally, let

$$h(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 2 - x & \text{if } x > 0 \end{cases}.$$

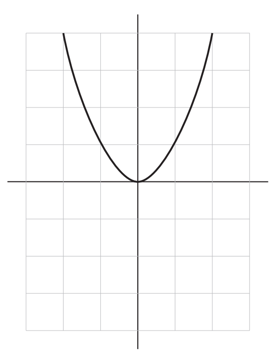
The graph of h looks like this:



□

The **range** of a function $y = f(x)$ is the set of possible outputs (y-values, or heights).

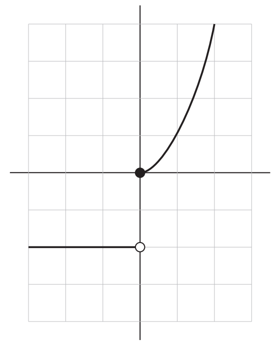
Example. Consider the function $f(x) = x^2$.



The graph attains every y-value greater than or equal to 0. Therefore, the range is $y \geq 0$. $\square$

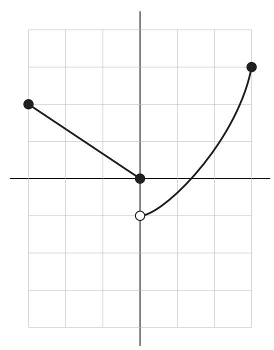
Example. Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0 \end{cases}.$$



The graph attains every y-value greater than or equal to 0 together with $y = -2$. The range is $y \geq 0$ and $y = 2$. $\square$

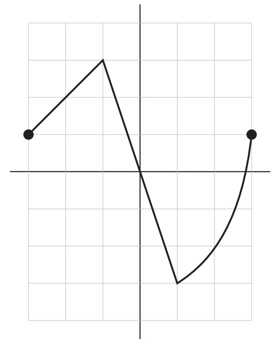
Example. Consider the function *whose entire graph* is shown below.



The function attains every y-value between -1 and 3. It attains $y = 3$ — the circle at $(3, 3)$ is filled in, so there *is* a point there. However, it does not attain the value -1, since the circle at $(0, -1)$ is open, meaning that the point is missing. The range is $-1 < y \leq 3$. $\square$

A graph is **increasing** if it goes up from left to right; it is **decreasing** if it goes down from left to right.

Example. Consider the function *whose entire graph* is shown below.

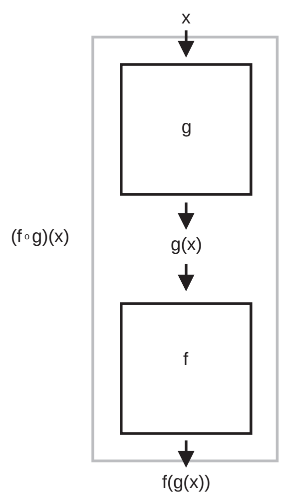


The function increases on the intervals $-3 \le x \le -1$ and on $1 \le x \le 3$. It decreases on the interval $-1 \le x \le 1$. □

If f and g are functions, their **composite** is

$$(f \circ g)(x) = f(g(x)).$$

The $\circ$ does *not* mean multiplication; it means that g is "inside" of f, or that the output of g is fed into f.



Example. Let $f(x) = x^3$ and let $g(x) = \dfrac{x}{x+1}$. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$, and $(g \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{x+1}\right) = \left(\frac{x}{x+1}\right)^3.$$

$$(g \circ f)(x) = g(f(x)) = g(x^3) = \frac{x^3}{x^3+1}.$$

$$(f \circ f)(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9.$$

$$(g \circ g)(x) = g\left(\frac{x}{x+1}\right) = \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1}.$$

Notice that $(f \circ g)(x)$ is not the same as $(g \circ f)(x)$. □

Example. Let $f(x) = x + 2$ and let $g(x) = 3x$. Find $f(g(5))$, $g(f(5))$, $f(g(x))$, and $g(f(x))$.

$$f(g(5)) = f(3 \cdot 5) = f(15) = 15 + 2 = 17.$$

$$g(f(5)) = g(5 + 2) = g(7) = 3 \cdot 7 = 21.$$

$$f(g(x)) = f(3x) = 3x + 2.$$

$$g(f(x)) = g(x + 2) = 3(x + 2).$$

Notice that $f(g(x))$ is not the same as $g(f(x))$. □
