Functions and Graphs

A function is a rule which assigns a unique output to each input.

Example. In mathematics, functions are often denoted using notation like the following:

\[ f(x) = x^2. \]

This says that the name of the function is \( f \), and \( x \) denotes a typical input. The right side tells how to produce an output from the input: Square it. Thus,

\[ f(1) = 1^2 = 1, \quad f(5) = 5^2 = 25, \quad \text{but note } f(-5) = (-5)^2 = 25 \quad \text{as well.} \]

Thus, different inputs can produce the same output.

Here is a table of some values of \( f \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2.1</th>
<th>-3</th>
<th>( \frac{1}{2} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>4.41</td>
<td>9</td>
<td>( \frac{1}{4} )</td>
<td>( \pi^2 )</td>
</tr>
</tbody>
</table>

Similarly,

\[ f(x + 1) = (x + 1)^2, \quad \text{and} \quad f(a + b) = (a + b)^2. \]

Likewise, suppose \( g(x) = \frac{1}{x+3} \). Then

\[ g(2) = \frac{1}{5}, \quad g(-17) = -\frac{1}{14}, \quad \text{but} \quad g(-3) \quad \text{is undefined.} \]

Moreover,

\[ g(x^2) = \frac{1}{x^2 + 3}, \quad \text{and} \quad g(x - 1) = \frac{1}{(x - 1) + 3} = \frac{1}{x + 2}. \]

The domain of a function is the set of allowable inputs.

Example. Suppose \( g(x) = \frac{1}{x+3} \). I can’t plug in \( x = -3 \), as I saw above. Any other number may be plugged in for \( x \). The domain consists of all \( x \) except \( x = 3 \) — sometimes I’ll get lazy and write this as “\( x \neq 3 \)” for short. □

Example. Suppose \( h(x) = \frac{x}{x^2 - x - 2} \). Then

\[ h(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x - 2)(x + 1)}. \]

Plugging in \( x = 2 \) or \( x = -1 \) would cause division by 0; other values of \( x \) are okay. Hence, the domain consists of all real numbers except for 2 and -1. □
You can graph a function by making a table of some function values, then plotting the points.

**Example.** Here’s the graph of $f(x) = x + 3$:

Here’s the graph of $y = |x|$:  

Here’s the graph of $x = y^2$:  

$x = y^2$ is not a function, since some inputs produce more than one output. For example, $x = 1$ gives $y = 1$ or $y = -1$.

*A graph represents a function if and only if every vertical line hits the graph at most once.*
Here’s the graph of $x^2 + y^2 = 1$:

This is also not the graph of a function.

Finally, let

\[ h(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 2 - x & \text{if } x > 0 \end{cases}. \]

The graph of $h$ looks like this:

The **range** of a function $y = f(x)$ is the set of possible outputs ($y$-values, or heights).

**Example.** Consider the function $f(x) = x^2$. 
The graph attains every $y$-value greater than or equal to 0. Therefore, the range is $y \geq 0$. □

**Example.** Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0 \end{cases}.$$  

The graph attains every $y$-value greater than or equal to 0 together with $y = -2$. The range is $y \geq 0$ and $y = 2$. □

**Example.** Consider the function whose entire graph is shown below.

The function attains every $y$-value between $-1$ and $3$. It attains $y = 3$ — the circle at $(3, 3)$ is filled in, so there is a point there. However, it does not attain the value $-1$, since the circle at $(0, -1)$ is open, meaning that the point is missing. The range is $-1 < y \leq 3$. □

A graph is **increasing** if it goes up from left to right; it is **decreasing** if it goes down from left to right.
Example. Consider the function whose entire graph is shown below.

The function increases on the intervals \(-3 \leq x \leq -1\) and on \(1 \leq x \leq 3\). It decreases on the interval \(-1 \leq x \leq 1\). 

If \(f\) and \(g\) are functions, their composite is
\[
(f \circ g)(x) = f(g(x)).
\]
The \(\circ\) does not mean multiplication; it means that \(g\) is “inside” of \(f\), or that the output of \(g\) is fed into \(f\).

Example. Let \(f(x) = x^3\) and let \(g(x) = \frac{x}{x + 1}\). Find \((f \circ g)(x)\), \((g \circ f)(x)\), \((f \circ f)(x)\), and \((g \circ g)(x)\).

\[
(f \circ g)(x) = f(g(x)) = f \left( \frac{x}{x + 1} \right) = \left( \frac{x}{x + 1} \right)^3.
\]
\[
(g \circ f)(x) = g(f(x)) = g(x^3) = \frac{x^3}{x^3 + 1}.
\]
\[
(f \circ f)(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9.
\]
\[
(g \circ g)(x) = g \left( \frac{x}{x + 1} \right) = \frac{x}{x + 1} + 1.
\]
Notice that \((f \circ g)(x)\) is not the same as \((g \circ f)(x)\). □

**Example.** Let \(f(x) = x + 2\) and let \(g(x) = 3x\). Find \(f(g(5))\), \(g(f(5))\), \(f(g(x))\), and \(g(f(x))\).

\[
\begin{align*}
  f(g(5)) & = f(3 \cdot 5) = f(15) = 15 + 2 = 17. \\
  g(f(5)) & = g(5 + 2) = g(7) = 3 \cdot 7 = 21. \\
  f(g(x)) & = f(3x) = 3x + 2. \\
  g(f(x)) & = g(x + 2) = 3(x + 2).
\end{align*}
\]

Notice that \(f(g(x))\) is not the same as \(g(f(x))\). □