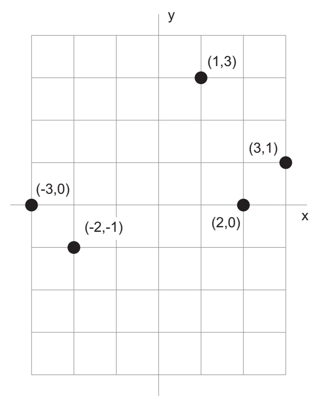
Cartesian Coordinates and Graphing

An **ordered pair** (x, y) of real numbers can be represented by a point in the plane. Construct a pair of perpendicular lines, one horizontal (the x**-axis**), the other vertical (the y**-axis**). Locate the point on the plane corresponding to (x, y) by starting at the **origin** — the place where the axes cross — and moving x units horizontally and y units vertically. If x is positive, you move to the right; if x is negative, you move to the left. Likewise, if y is positive, you move up; if y is negative, you move down.

x and y are the **Cartesian coordinates** of the point.

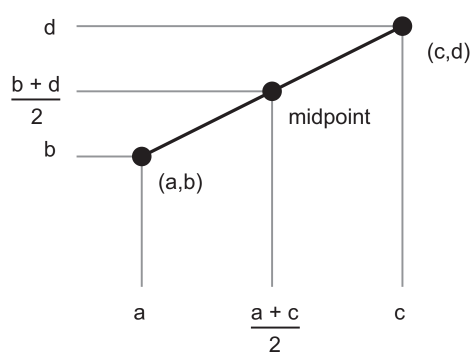
Example. Plot the points $(1, 3)$, $(3, 1)$, $(4, 0)$, $(0, -5)$, and $(-2, -1)$ on a set of coordinate axes.



□

The **midpoint** of a segment is the point halfway between the two points. If the points are (a, b) and (c, d), the midpoint is

$$\left(\frac{1}{2}(a+c), \frac{1}{2}(b+d)\right).$$



Example. Find the midpoint of the segment joining the points $(2, 5)$ and $(-3, 7)$.

$$\frac{1}{2}(2 + (-3)) = -\frac{1}{2} \quad \text{and} \quad \frac{1}{2}(5 + 7) = 6.$$

The midpoint is $\left(-\frac{1}{2}, 6\right)$. □

The **distance** between points (a, b) and (c, d) can be found using Pythagoras' theorem. It is

$$d = \sqrt{(a - c)^2 + (b - d)^2}.$$

Example. Find the distance between the points $(3, -7)$ and $(15, -12)$.

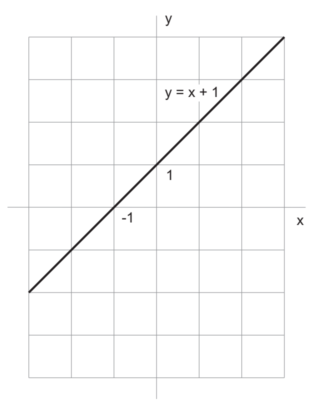
$$d = \sqrt{(3 - 15)^2 + (-7 - (-12))^2} = \sqrt{(-12)^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13. \quad \square$$

An equation involving x and y can be represented by a set of points in the plane — the **graph** of the equation. The graph of an equation consists of all points whose coordinates (x, y) satisfy the equation.

Example. Graph $y = x + 1$.

Select some x-values and plug them into the equation to find the corresponding y-values. Then plot the resulting points.

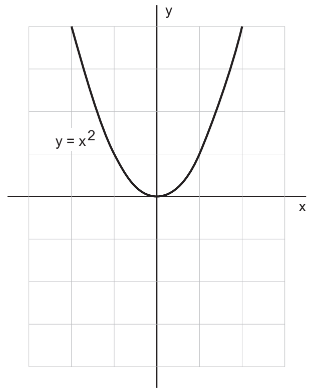
x	-3	-2	-1	0	1	2	3
y = x + 1	-2	-1	0	1	2	3	4



$\square$

Example. Graph $y = x^2$.

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4



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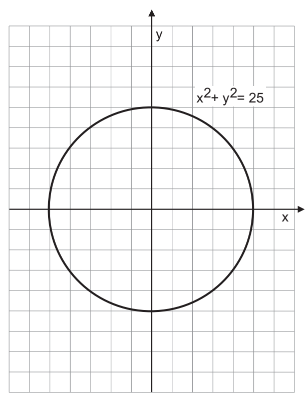
Example. Graph $x^2 + y^2 = 25$.



□

The x**-intercepts** of an equation are the places where the graph crosses the x-axis. You can find the x-intercepts by setting $y = 0$ and solving for x.

The y**-intercepts** of an equation are the places where the graph crosses the y-axis. You can find the x-intercepts by setting $x = 0$ and solving for y.

Example. Find the x-intercepts and y-intercepts of $y = x^2 - 3x - 4$.

To find the y-intercepts, set $x = 0$:

$$y = 0^2 - 3 \cdot 0 - 4 = -4.$$

To find the x-intercepts, set $y = 0$:

$$0 = x^2 - 3x - 4, \quad 0 = (x - 4)(x + 1), \quad x = 4 \quad \text{or} \quad x = -1. \quad \square$$

Example. Find the x-intercepts and y-intercepts of $y = \dfrac{x + 3}{x - 4}$.

To find the y-intercepts, set $x = 0$:

$$y = \frac{0 + 3}{0 - 4} = -\frac{3}{4}.$$

To find the x-intercepts, set $y = 0$:

$$0 = \frac{x+3}{x-4}, \quad (x-4)\cdot 0 = (x-4)\cdot\frac{x+3}{x-4}, \quad 0 = x+3, \quad x = -3. \quad \Box$$

In what follows, I'll say two equations are **the same** if you can get from either one to the other using valid algebra.

You can often use **symmetry** as an aid in graphing equations.

The graph of an equation is **symmetric about the y-axis** if the equation is the same when x is replaced with $-x$.

Example. $y = \dfrac{1}{x^2}$ is symmetric about the y-axis. If I replace x with $-x$, I get

$$y = \frac{1}{(-x)^2} = \frac{1}{x^2},$$

which is the same as the original equation.

On the other hand, consider $y = x - 5$. If I replace x with $-x$, I get

$$y = (-x) - 5 = -x - 5.$$

This is not the same as the original equation, so the graph is not symmetric about the y-axis. $\Box$

The graph of an equation is **symmetric about the x-axis** if the equation is the same when y is replaced with $-y$.

Example. $x = y^2 + 5$ is symmetric about the x-axis. If I replace y with $-y$, I get

$$x = (-y)^2 + 5 = y^2 + 5,$$

which is the same as the original equation.

On the other hand, consider $x + y = 3$. If I replace y with $-y$, I get

$$x + (-y) = 3 \quad \text{or} \quad x - y = 3.$$

This is not the same as the original equation, so the graph is not symmetric about the x-axis. $\Box$

The graph of an equation is **symmetric about the origin** if the equation is the same when x is replaced with $-x$ and y is replaced with $-y$.

Example. $xy = 1$ is symmetric about the origin. If I replace x with $-x$ and y with $-y$, I get

$$(-x)(-y) = 1 \quad \text{or} \quad xy = 1.$$

This is the same as the original equation.

On the other hand, consider $y = x^2$. If I replace x with $-x$ and y with $-y$, I get

$$-y = (-x)^2 \quad \text{or} \quad -y = x^2.$$

This is not the same as the original equation, so the graph is not symmetric about the origin. □

An equation of the form

$$(x-a)^2 + (y-b)^2 = r^2$$

represents a **circle** of radius r whose center is the point (a, b).

Example. The equation

$$(x-4)^2 + (y+17)^2 = 25$$

represents a circle of radius $\sqrt{25} = 5$ with center $(4, -17)$. □

Example. Find an equation for the circle of radius 3 whose center is $(7, -10)$.

$$(x-7)^2 + (y+10)^2 = 9. \quad □$$

Example. Find the radius and the center of the circle $x^2 + y^2 - 6y = 55$.

I need to add a number to $y^2 - 6y$ to make a perfect square.

$$\frac{-6}{2} = -3 \quad \text{and} \quad (-3)^2 = 9,$$

so I add 9 to both sides:

$$x^2 + y^2 - 6y = 55, \quad x^2 + y^2 - 6y + 9 = 55 + 9, \quad x^2 + (y-3)^2 = 64.$$

Since $\sqrt{64} = 8$, the radius is 8. The center is $(0, 3)$. □

Example. Find the radius and the center of the circle $x^2 + 2x + y^2 - 10y = 10$.

I need to add numbers to $x^2 + 2x$ and to $y^2 - 10y$ to make perfect squares.

$$\frac{2}{2} = 1 \quad \text{and} \quad 1^2 = 1.$$

$$\frac{-10}{2} = -5 \quad \text{and} \quad (-5)^2 = 25.$$

Thus, I need to add 1 to the first expression and 25 to the second. Since I'm adding $1 + 25 = 26$ to the left side, I must add 26 to the right side of the equation as well.

$$x^2 + 2x + y^2 - 10y = 10, \quad x^2 + 2x + 1 + y^2 - 10y + 25 = 10 + 26, \quad (x+1)^2 + (y-5)^2 = 36.$$

The center is $(-1, 5)$ and the radius is 6. □

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