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Integer Exponents

I'll discuss some rules for working with **integer exponents**. Actually, these rules work with arbitrary exponents, but it is easier to explain why they're true in this case. So in what follows, all the powers are assumed to be positive or negative integers.

$$a^m \cdot a^n = a^{m+n}.$$

Let's do a specific example to check that this makes sense.

$$a^3 \cdot a^4 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^7.$$

Negative exponents translate to reciprocals. That is,

$$a^{-n} = \frac{1}{a^n}.$$

So

$$a^{-3} = \frac{1}{a^3}, \qquad \text{and} \qquad \frac{1}{x^{42}} = x^{-42}.$$

Here is a rule for raising a power to a power:

$$(a^m)^n = a^{mn}.$$

Again, I'll do a specific example to check that this makes sense.

$$(a^3)^2 = a^3 \cdot a^3 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) = a^6.$$

Now I'll use these rules to simplify some expressions.

Example.

$$3^6 \cdot 3^8 = 3^{14}, \quad \text{but} \quad (3^6)^8 = 3^{48}.$$

Warning: $-3^2 = -9$. The expression "-3^2" is read to mean that you square the 3 *first*, then negate the result. On the other hand, $(-3)^2 = 9$.

A number raised to an even power is always positive:

$$\left((-3)^2\right)^5 = (-3)^{10} = 59049.$$

On the other hand, a *negative* number raised to an odd power is negative:

$$(-5)^3 = -125. \quad \square$$

Example.

$$x^7 x^{-5} = x^2.$$

$$(x^4)^{-3} = x^{-12} = \frac{1}{x^{12}}.$$

(Whether you write x^{12} or $\frac{1}{x^{12}}$ depends on what you're using the expression for.)

$$\left(\frac{1}{x^2}\right)^4 = (x^{-2})^4 = x^{-8} = \frac{1}{x^8}. \quad \square$$

By the way, these rules work with non-integer exponents. For example,

$$x^{1/3}x^{3/2} = x^{11/6}, \quad \text{because} \quad \frac{1}{3} + \frac{3}{2} = \frac{11}{6}.$$

$$(ab)^n = a^n b^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Here's a specific case of the first rule so you can see that it makes sense.

$$(ab)^3 = (ab)(ab)(ab) = (aaa)(bbb) = a^3b^3.$$

Example. Simplify and write the result using positive powers.

$$(x^2y^3)^4 = (x^2)^4(y^3)^4 = x^8y^{12}.$$

$$(-2x^5y)^7 = (-2)^7(x^5)^7y^7 = -128x^{35}y^7.$$

$$2(3x^6y)^2x^4y^{-5} = 2(3^2)(x^6)^2y^2x^4y^{-5} = 2\cdot 9x^{12}y^2x^4y^{-5} = 18x^{16}y^{-3} = \frac{18x^{16}}{y^3}.$$

$$\left(\frac{2x^2}{y^6}\right)^4 = \frac{2^4(x^2)^4}{(y^6)^4} = \frac{16x^8}{y^{24}}.$$

$$(2x^3y^{-5})^{-2} = 2^{-2}(x^3)^{-2}(y^{-5})^{-2} = \frac{1}{2^2}x^{-6}y^{10} = \frac{1}{4}\cdot\frac{1}{x^6}\cdot y^{10} = \frac{y^{10}}{4x^6}. \quad \square$$

You can derive a rule for division from the rule for multiplication:

$$\frac{a^m}{a^n} = a^m \cdot a^{-n} = a^{m-n}.$$

Example.

$$\frac{2^9}{2^3} = 2^6.$$

$$\frac{(x+2)^{13}}{(x+2)^{17}} = (x+2)^{-4}.$$

$$\frac{y^{3n}}{y^n} = y^{2n}, \quad \text{since} \quad 3n - n = 2n.$$

But

$$\frac{a^4}{b^6} \quad \text{can't be simplified.}$$

The things being raised to powers (a and b) are not the same, so the rules I've given don't apply. □

Example. Simplify, writing the result in terms of positive powers:

$$\frac{(x^2y^3)^4}{x^5(y^2)^7} = \frac{x^8y^{12}}{x^5y^{14}} = \frac{x^3}{y^2}. \quad \square$$

Example. Simplify, writing the result in terms of positive powers:

$$(4x^3y^4z^{-5})^2 = 16x^6y^8z^{-10} = \frac{16x^6y^8}{z^{10}}. \quad \square$$

Example. Simplify, writing the result in terms of positive powers:

$$(3x^5y^{-7})^4 = 3^4(x^5)^4(y^{-7})^4 = 81x^{20}y^{-28} = \frac{81x^{20}}{y^{28}}. \quad \square$$

Example.

$$\left(\frac{5}{3}\cdot\frac{a^2}{b^3}\right)^4 = \frac{625a^8}{81b^{12}}.$$

$$(a^2b^3)^{4n} = a^{8n}b^{12n}. \quad \square$$

Example. Simplify, writing the result in terms of positive powers:

$$\frac{(x^{-2}y^3)^5}{(x^2y^{-3})^4} = \frac{x^{-10}y^{15}}{x^8y^{-12}} = x^{-18}y^{27} = \frac{y^{27}}{x^{18}}. \quad \square$$

Example. Simplify, writing the result in terms of positive powers:

$$\frac{(2ab^{-2})^{-3}}{(4a^2b^3)^2} = \frac{2^{-3}a^{-3}b^6}{16a^4b^6} = \frac{1}{128}a^{-7} = \frac{1}{128a^7}. \quad \square$$
