7-29-2008

Inverse Functions

Functions $f : X \to Y$ and $g : Y \to X$ are **inverses** if

$$f(g(y)) = y \qquad \text{and} \qquad g(f(x)) = x$$

for all $x \in X$ and $y \in Y$. If f has an inverse, it is often denoted f^{-1}. However, f^{-1} *does not mean* "$\dfrac{1}{f}$"!

Example. $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses, since

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x \qquad \text{and} \qquad g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

if x is a real number.

Notice that $f(x) = x^3$ but the inverse is *not* $\dfrac{1}{x^3}$! □

Example. Functions which are inverses "undo" one another. Thus, if f and f^{-1} are inverses and f takes 4 to 17, then f^{-1} must take 17 to 4.

In symbols,

$$f(4) = 17 \quad \text{implies} \quad f^{-1}(17) = 4. \quad \square$$

Example. In some cases, it's possible to find the inverse of a function algebraically. I'll find the inverse of $f(x) = x^3 + 5$.

First, I'll write it as $y = x^3 + 5$.

Next, switch x's and y's:

$$x = y^3 + 5.$$

(This means you should replace each "x" with a "y" and replace each "y" with an "x".)

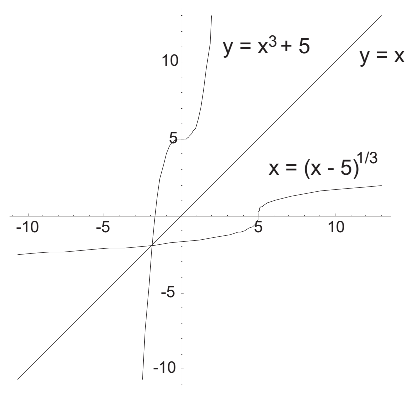
Now solve for y in terms of x:

$$\begin{aligned} x &= y^3 + 5 \\ x - 5 &= y^3 + 5 - 5 \\ x - 5 &= y^3 \\ (x-5)^{1/3} &= (y^3)^{1/3} \\ (x-5)^{1/3} &= y \end{aligned}$$

Since I was able to solve for y in terms of x, the result is the inverse function: $f^{-1}(x) = y = (x-5)^{1/3}$.

The procedure I used tells something about the relation between the graphs of a function and its inverse. Since the inverse is obtained by swapping x's and y's, the graph of f^{-1} is a *mirror image* of the graph of f across the line $y = x$.

In the picture below, I've shown the graphs of $f(x) = x^3 + 5$, $f^{-1}(x) = (x-5)^{1/3}$, and $y = x$:



□

Example. Find the inverse of $f(x) = \dfrac{1}{x+1}$.

Let $y = \dfrac{1}{x+1}$. Swap x's and y's to obtain

$$x = \frac{1}{y+1}.$$

Solve for y:

$$\begin{aligned} x(y+1) &= 1 \\ \frac{1}{x} \cdot x(y+1) &= \frac{1}{x} \cdot 1 \\ y + 1 &= \frac{1}{x} \\ y + 1 - 1 &= \frac{1}{x} - 1 \\ y &= \frac{1}{x} - 1 \end{aligned}$$

Therefore, $f^{-1}(x) = y = \dfrac{1}{x} - 1$. □

Example. Find the inverse of $f(x) = \dfrac{x-5}{x}$.

Let $y = \dfrac{x-5}{x}$. Swap x's and y's to obtain

$$x = \frac{y-5}{y}.$$

Solve for y:

$$\begin{aligned}
x &= \frac{y-5}{y} \\
y \cdot x &= y \cdot \frac{y-5}{y} \\
xy &= y - 5 \\
xy - y &= y - 5 - y \\
xy - y &= -5 \\
y(x-1) &= -5 \\
\frac{1}{x-1} \cdot y(x-1) &= \frac{1}{x-1} \cdot -5 \\
y &= \frac{-5}{x-1}
\end{aligned}$$

Hence, $f^{-1}(x) = y = \dfrac{-5}{x-1}$. □

Not every function has an inverse. For example, consider $f(x) = x^2$. Now $f(2) = 4$, so f^{-1} should take 4 back to 2. But $f(-2) = 4$ as well, so apparently f^{-1} should take 4 to -2. f^{-1} can't do both, so there is no inverse! The problem is that you can't undo the effect of the squaring function in a unique way.

On the other hand, if I restrict $f(x) = x^2$ to $x \ge 0$, then it has an inverse function: $f^{-1} = \sqrt{x}$.

A function f is **one-to-one** or **injective** if different inputs go to different outputs:

$$x \ne y \quad \text{implies} \quad f(x) \ne f(y).$$

Example. $f(x) = x^4$ is not one-to-one, because different inputs can produce the same output. For example,

$$f(3) = 3^4 = 81 \quad \text{and} \quad f(-3) = (-3)^4 = 81.$$

On the other hand, $g(x) = 3x + 4$ is one-to-one. For suppose the inputs a and b produce the same output: $f(a) = f(b)$. Then

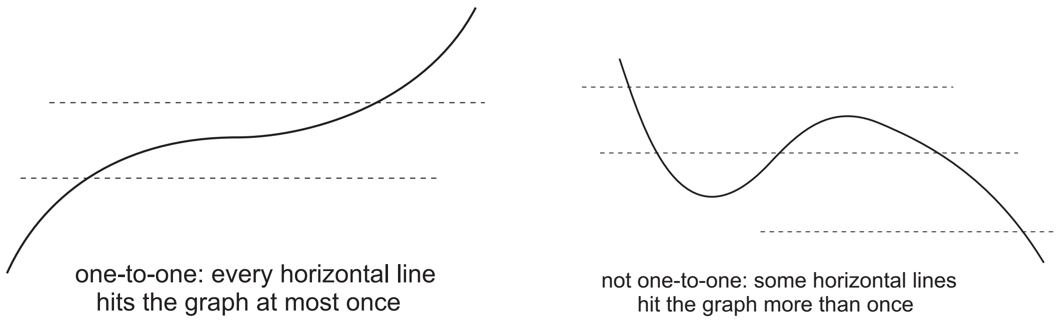
$$3a + 4 = 3b + 4a.$$

Then

$$\begin{array}{cccccc}
 & 3a & + & 4 & = & 3b & + & 4 \\
- & & & 4 & & & & 4 \\
\hline
 & 3a & & & = & 3b & & \\
/ & 3 & & & & 3 & & \\
\hline
 & a & & & = & b & &
\end{array}$$

That is, the inputs a and b were the same to begin with. □

A graph of a function represents a one-to-one function if every horizontal line hits the graph at most once.



A one-to-one function has an inverse: Since a given output could have only come from *one* input, you can undo the effect of the function.