8-13-2016

Linear Inequalities

Solving linear inequalities is very similar to solving equations; the only difference is that $=$ is replaced with $<$, $>$, $\leq$, or $\geq$. Here are some linear inequalities:

$$2x + 1 < -5$$

$$3x - 2 \geq 8x + 14$$

$$2 < 3x + 5 \leq 26$$

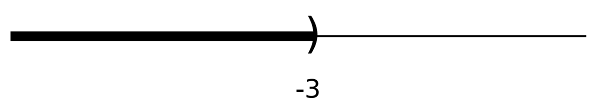
$$\frac{1}{2}x + 4 > \frac{2}{3}x + 6$$

The idea is the same as in solving linear equations: **Try to get all the variable terms on one side and all the number terms on the other**.

Example. Solve $2x + 1 < -5$.

$$\begin{array}{cccccc} & 2x & + & 1 & < & -5 \\ - & & & 1 & & 1 \\ \hline & 2x & & & < & -6 \\ \div & 2 & & & & 2 \\ \hline & x & & & < & -3 \end{array}$$

The solution is $x < -3$. The graph of the solution is shown below:



Notice that the parenthesis ("$)$") means that $x = -3$ is *not* included in the solution set.

You can also write solutions to inequalities using **interval notation**. In this case, the solution would be $(-\infty, -3)$. Notice that the "$)$" next to the "-3" matches the "$)$" on the picture.

In interval notation, (round) parentheses next to a number mean that the number *is not* included; (square) brackets next to a number mean that the number *is* included. I put a "$)$" next to the -3, because -3 *is not* included in the solution set. If the solution set had been $x \leq -3$, then -3 would be included, and the interval notation for the solution would be $(-\infty, -3]$. Notice that you *always* use parentheses next to $-\infty$ or ∞. $\square$

You have almost the same tools for inequality-solving as for equation-solving: roughly, you're allowed to perform the same operation on both sides of an inequality. But there is an important difference when it comes to multiplication or division.

1. You can add the same thing to both sides or subtract the same thing from both sides.

2. You can multiply or divide both sides by the same thing. But you can't multiply or divide by 0. **Note:** if you multiply or divide by a negative number, the inequality reverses direction.

Why does multiplying an inequality by a negative number reverse the inequality? Try it with numbers. This inequality is true:

$$8 > -3.$$

Multiply both sides by -2:

$$-16 \ (?) \ 6.$$

Which way should the inequality go?

If I leave it as it was, the new inequality would be "$-16 > 6$", which is *wrong*.

If I reverse the inequality, the new inequality would be "$-16 < 6$", which is *right*.

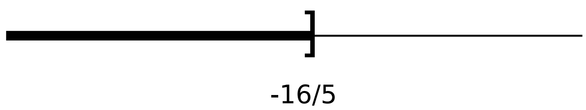
You can see by this example that if you multiply both sides of an inequality by a negative number, you should reverse the inequality.

Example. Solve $3x - 2 \geq 8x + 14$.

$$\begin{array}{cccccccc} & 3x & - & 2 & \geq & 8x & + & 14 \\ + & & & 2 & & & & 2 \\ \hline & 3x & & & \geq & 8x & + & 16 \\ - & 8x & & & & 8x & & \\ \hline & -5x & & & \geq & & & 16 \\ \div & -5 & & & & & & -5 \\ \hline & x & & & \leq & & & -\dfrac{16}{5} \end{array}$$

In the last step, I divided both sides by -5, a negative number. Therefore, I had to reverse the inequality.

The solution is $x \leq -\dfrac{16}{5}$, or $\left(-\infty, -\dfrac{16}{5}\right]$ in interval notation. The graph of the solution is shown below:



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Example. Solve $3(5x - 1) > 15x + 10$.

$$\begin{aligned} 3(5x - 1) &> 15x + 10 \\ 15x - 3 &> 15x + 10 \\ 15x - 3 - 15x &> 15x + 10 - 15x \\ -3 &> 10 \end{aligned}$$

The last inequality is a **contradiction**, because -3 is not bigger than 10. Hence, the original inequality has no solutions. □

Note: You could write your answer as $\emptyset$. Don't use (nonstandard) abbreviations (such as "DNE".)

Example. Solve $4(1 - 3x) < 6(5 - 2x)$.

$$\begin{aligned} 4(1 - 3x) &< 6(5 - 2x) \\ 4 - 12x &< 30 - 12x \\ 4 - 12x + 12x &< 30 - 12x + 12x \\ 4 &< 30 \end{aligned}$$

The last equation is an **identity**, because it's true that 4 is less than 30. Hence, the solution to the original inequality is all real numbers. □

Note: You could write your answer as $\mathbb{R}$, or as $(-\infty, \infty)$,

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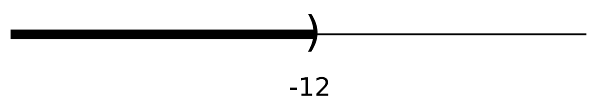
Example. Solve $\dfrac{1}{2}x + 4 > \dfrac{2}{3}x + 6$.

In this case, it's useful to clear the fractions first. The denominators of the fractions are 2 and 3. So I'll multiply both sides by 6, which is the least common multiple of 2 and 3.

$$\begin{aligned}\frac{1}{2}x + 4 &> \frac{2}{3}x + 6 \\ 6 \cdot \left(\frac{1}{2}x + 4\right) &> 6 \cdot \left(\frac{2}{3}x + 6\right) \\ 6 \cdot \frac{1}{2}x + 6 \cdot 4 &> 6 \cdot \frac{2}{3}x + 6 \cdot 6 \\ 3x + 24 &> 4x + 36 \\ 3x + 24 - 3x - 36 &> 4x + 36 - 3x - 36 \\ -12 &> x\end{aligned}$$

The solution is $x < -12$, or $(-\infty, -12)$ in interval notation. The graph of the solution set is shown below:



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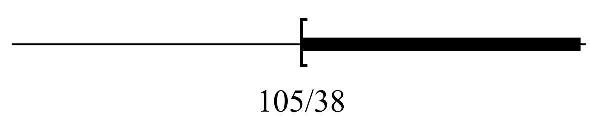
Example. Solve $0.2 + 0.13x \le 0.17(3x - 5)$.

In this case, I'll begin by clearing the decimals. In the decimals 0.2, 0.13, and 0.17, the largest number of places to the right of the decimal point is 2. So the clear the decimals, I need to multiply both sides by $10^2 = 100$.

$$\begin{aligned}0.2 + 0.13x &\le 0.17(3x - 5) \\ 100 \cdot (0.2 + 0.13x) &\le 100 \cdot 0.17(3x - 5) \\ 100 \cdot 0.2 + 100 \cdot 0.13x &\le 100 \cdot 0.17(3x - 5) \\ 20 + 13x &\le 17(3x - 5) \\ 20 + 13x &\le 51x - 85 \\ 20 + 13x - 20 - 51x &\le 51x - 85 - 20 - 51x \\ -38x &\le -105 \\ \frac{-38x}{-38} &\ge \frac{-105}{-38} \\ x &\ge \frac{105}{38}\end{aligned}$$

Note that I had to flip the inequality when I divided both sides by -38.

The solution is $x \ge \dfrac{105}{38}$, or $\left[\dfrac{105}{38}, \infty\right)$. The graph of the solution set is shown below:



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Example. (**A compound inequality**) Solve $2 < 3x + 5 \le 26$.

$$\begin{array}{rrcrcrcr} & 2 & < & 3x & + & 5 & \le & 26 \\ - & 5 & < & 3x & & 5 & & 5 \\ \hline & -3 & < & 3x & & & \le & 21 \\ \div & 3 & & 3 & & & & 3 \\ \hline & -1 & < & x & & & \le & 7 \end{array}$$

The solution is $-1 < x \le 7$. In interval notation, this is $(-1, 7]$. The graph of the solution set is shown below:

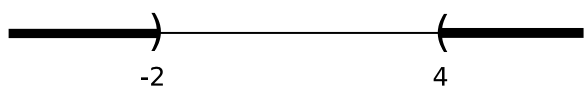
□

Example. (**A two-part inequality**) Solve the two-part inequality $3x < -6$ or $5x > 20$.

In this case, I just solve the inequalities separately.

$$\begin{array}{rrcr} & 3x & < & -6 \\ \div & 3 & & 3 \\ \hline & x & < & -2 \end{array} \qquad \begin{array}{rrcr} & 5x & > & 20 \\ \div & 5 & & 5 \\ \hline & x & > & 4 \end{array}$$

The solution is $x < -2$ or $x > 4$. In interval notation, this is $(-\infty, -2) \cup (4, \infty)$. (The "$\cup$" is the symbol for the **union** of the two sets.) The graph of the solution set is shown below:



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