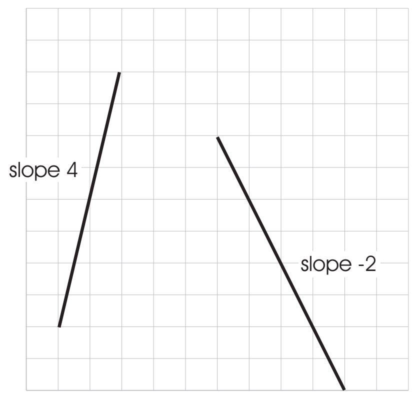
Lines

The **slope** of the line which passes through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

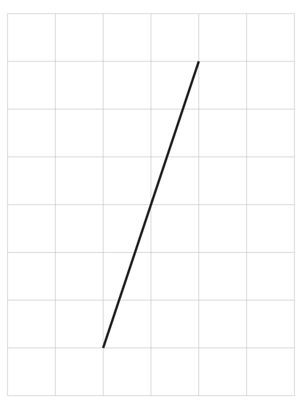
The slope measures the *rate* at which a line goes up or down as you move to the right. For example, a line with slope 4 goes *up* 4 units for every 1 unit you move to the right. A line with slope -2 goes *down* 2 units for every 1 unit you move to the right.



Example. Find the slope of the line which passes through $(1, 4)$ and $(-5, 3)$.

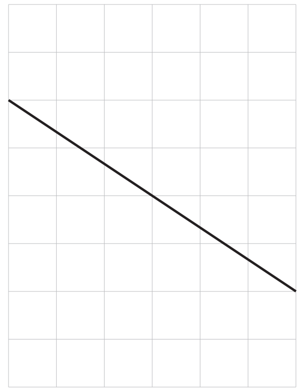
$$\frac{3-4}{-5-1} = \frac{1}{6}. \quad \square$$

Example. A line has slope 3. If you move 2 units to the right, how far up or down does the line go?



$$2 \cdot 3 = 6. \quad \square$$

Example. A line has slope $-\frac{2}{3}$. Does it go from northwest to southeast or from southwest to northeast?



It goes from northwest to southeast. □

Example. Find the slope of the line which passes through $(0, 2)$ and $(17, 2)$.

$$m = \frac{2-2}{17-0} = 0$$

A line with slope 0 is *horizontal* — parallel to the x-axis. □

Example. Find the slope of the line which passes through $(2, 0)$ and $(2, 17)$.

If I use the slope formula, I get

$$m = \frac{17-0}{2-2} = \frac{17}{0} \quad \text{(undefined)}$$

A vertical line has *undefined* slope. □

A line may be represented by various equations. Here are a few important forms:

$$ax + by = c, \text{ where } a, b, \text{ and } c \text{ are numbers}$$

$$y - y_0 = m(x - x_0), \text{ where } m, x_0, \text{ and } y_0 \text{ are numbers}$$

$$y = mx + b, \text{ where } m \text{ and } b \text{ are numbers}$$

You can get from one form to another using algebra. For example:

$$y - y_0 = m(x - x_0), \quad y - y_0 = mx - mx_0, \quad -mx + y = -mx_0 + y_0.$$

The last equation is in $ax + by = c$ form, with $a = -m$, $b = 1$, and $c = -mx_0 + y_0$.

Example. Which of the following equations represent lines?

(a) $4x - 3y = 17$

(b) $\frac{1}{2}x = 5 + 3y$

(c) $x^2 + 3y = 42$

(d) $\frac{y}{x} = 6$

$4x - 3y = 17$ is a line, as is $\frac{1}{2}x = 5 + 3y$.

$x^2 + 3y = 42$ is not a line, because of the x^2 term.

$\frac{y}{x} = 6$ is almost a line — in fact, if I multiply both sides by x, I get $y = 6x$, which is a line. But the original equation is a line minus one point, because plugging in $x = 0$ would cause division by 0. □

Consider the form

$$y - y_0 = m(x - x_0), \text{ where } m, x_0, \quad \text{and} \quad y_0 \text{ are numbers}$$

This is called **point-slope form**. It is the equation of a line with slope m which passes through the point (x_0, y_0).

Example. Find the equation of the line with slope -17 which passes through the point $(3, 5)$.

$$y - 5 = -17(x - 3) \quad \square$$

Example. Find the slope of the line $y - 16 = 3(x + 4)$. Find at least two different points on the line.

The slope is $m = 3$. And from the fact that the equation is in point-slope form, I can see that the line passes through the point $(-4, 16)$.

To get another point on the line, plug in a random value for x and solve for y. For example, if I use $x = 1$, I get

$$y - 16 = 3(1 + 4), \quad y - 16 = 15, \quad y = 31.$$

Therefore, the point $(1, 31)$ also lies on the line. □

Example. Find the equation of the line which passes through the points $(4, 2)$ and $(7, -10)$.

First, I'll find the slope:

$$m = \frac{-10 - 2}{7 - 4} = -4.$$

Using the point $(4, 2)$ and point-slope form, I get the equation

$$y - 2 = -4(x - 4).$$

If I used the point $(7, -10)$, I'd get

$$y + 10 = -4(x - 7).$$

In fact, the two forms are equivalent:

$$y - 2 = -4(x - 4), \quad y - 2 = -4x + 16, \quad y = -4x + 18$$

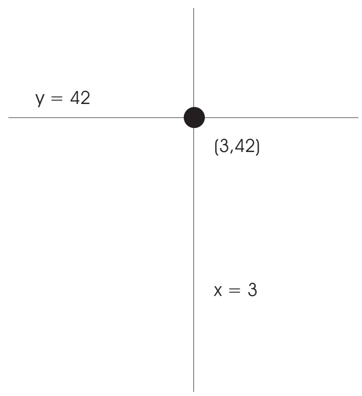
$$y + 10 = -4(x - 7), \quad y + 10 = -4x + 28, \quad y = -4x + 18 \quad \square$$

Example. Find the equation of the horizontal line which passes through the point $(3, 42)$. Find the equation of the vertical line which passes through the point $(3, 42)$.

A horizontal line has slope 0, so the equation of the horizontal line which passes through the point $(3, 42)$ is

$$y - 42 = 0 \cdot (x - 3), \quad y - 42 = 0, \quad y = 42.$$

A vertical line has undefined slope. But if you draw the picture, you can see that the vertical line passing through $(3, 42)$ is $x = 3$:



This works in general. For example, the horizontal line passing through $(-17, 8)$ is $y = 8$. The vertical line passing through $(-17, 8)$ is $x = -17$. $\square$

The y**-intercept** of a line is the point where the line intersects the y-axis. Likewise, the x**-intercept** is the point where the line intersects the x-axis.

The form

$$y = mx + b, \text{ where } m \quad \text{and} \quad b \text{ are numbers}$$

is called **slope-intercept form**. In this equation, m is the slope of the line and b is the y-intercept.

Example. Find the equation of the line with slope 15 and y-intercept -32.

$$y = 15x + 32 \quad \square$$

Example. Find the slope and y-intercept of the line $y = -17x$.

The slope is -17, while the y-intercept is $y = 0$. $\square$

Example. Find the slope and y-intercept of the line $3x + 4y = 24$.

I use algebra to put the equation into slope-intercept form:

$$3x + 4y = 24, \quad 4y = -3x + 24, \quad y = -\frac{3}{4}x + 6.$$

The slope is $m = -\frac{3}{4}$ and the y-intercept is $y = 6$. □

Example. Find the point where the lines $3x + y = 10$ and $2x - y = 5$ intersect.

To find where two graphs intersect, solve their equations simultaneously.
From $3x + y = 10$, I get $y = 10 - 3x$. Plug this into $2x - y = 5$:

$$2x - (10 - 3x) = 5, \quad 2x - 10 + 3x = 5, \quad 5x - 10 = 5, \quad 5x = 15, \quad x = 3.$$

Then $y = 10 - 3x = 10 - 3 \cdot 3 = 1$. The point of intersection is $(3, 1)$. □

Example. Find the x-intercept of the line $2x - 5y = 10$.

To find the x-intercept, set $y = 0$ and solve for x. The reason this works is that $y = 0$ is the x-axis, and the x-intercept is the place where the line intersects the x-axis. By the rule of thumb from the last problem, I therefore solve $2x - 5y = 10$ and $y = 0$ simultaneously:

$$2x - 5y = 10, \quad 2x - 5 \cdot 0 = 10, \quad 2x = 10, \quad x = 5.$$

The x-intercept is $x = 5$.
In the same way, *you can find the y-intercept by setting $x = 0$ and solving for y.* □

- Two lines are **parallel** if and only if they have the same slope.
- Two lines are **perpendicular** if their slopes are negative reciprocals of one another.

For example, 5 and $-\frac{1}{5}$ are negative reciprocals of one another. So are $\frac{3}{2}$ and $-\frac{2}{3}$. Two numbers are negative reciprocals if their product is -1:

$$\frac{4}{22} \cdot \left(-\frac{11}{2}\right) = -\frac{44}{44} = -1 \quad \text{They're negative reciprocals}$$

$$\frac{36}{8} \cdot \left(-\frac{2}{3}\right) = -\frac{72}{24} = -3 \quad \text{They aren't negative reciprocals}$$

Of course, any horizontal line is perpendicular to any vertical line.

Example. Determine whether the following lines are parallel, perpendicular, or neither:

$$3x - 4y = 12 \qquad \text{and} \qquad 8x + 6y = 11$$

I find the slopes by putting the lines into slope-intercept form:

$$3x - 4y = 12, \quad -4y = -3x + 12, \quad y = \frac{3}{4}x - 3$$

$$8x + 6y = 11, \quad 6y = -8x + 11, \quad y = -\frac{8}{6}x + \frac{11}{6}$$

The slopes of the lines are $\frac{3}{4}$ and $-\frac{8}{6}$. The slopes aren't equal, so the lines aren't parallel. Since

$$\frac{3}{4} \cdot -\frac{8}{6} = -1,$$

the slopes are negative reciprocals. Hence, the lines are perpendicular. □

Example. Find the equation of the line which passes through the point $(3, -4)$ and is parallel to the line $2x - 5y = 20$.

Find the slope of the given line:

$$2x - 5y = 20, \quad -5y = -2x + 20, \quad y = \frac{2}{5}x - 4.$$

The given line has slope $\frac{2}{5}$. The line I want is parallel to the given line, so it also has slope $\frac{2}{5}$. Since my line passes through the point $(3, -4)$, it has point-slope form

$$y + 4 = \frac{2}{5}(x - 3). \quad \square$$

Example. Find the equation of the line which is perpendicular to the line $x - 12y = 4$ and passes through the point $(7, 5)$.

Find the slope of the given line:

$$x - 12y = 4, \quad -12y = -x + 4, \quad y = \frac{1}{12}x - \frac{1}{3}.$$

The given line has slope $\frac{1}{12}$. The line I want is perpendicular to the given line, so my line must have slope -12 (the negative reciprocal of $\frac{1}{12}$). Since my line passes through $(7, 5)$, its equation is

$$y - 5 = -12(x - 7. \quad \square$$

Example. Find the equation of the line which passes through the point $(3, -1)$ and is parallel to the line which passes through $(2, 4)$ and $(5, -3)$.

The slope of the line which passes through $(2, 4)$ and $(5, -3)$ is

$$m = \frac{-3 - 4}{5 - 2} = -\frac{7}{3}.$$

The line I want is parallel to this line, so it also has slope $-\frac{7}{3}$. Since my line passes through $(3, -1)$, its point-slope form is

$$y + 1 = -\frac{7}{3}(x - 3). \quad \square$$
