7-29-2008

The Natural Logarithm

Let a be a positive number, $a \neq 1$, and let $x > 0$. The **logarithm of x to the base a** is the number $y = \log_a x$ such that $a^y = x$. That is,

$$y = \log_a x \quad \text{means} \quad a^y = x.$$

Example. What exponential equation is equivalent to $\log_2 16 = 4$?

$$\log_2 16 = 4 \quad \text{is equivalent to} \quad 2^4 = 16. \quad \square$$

Example. What logarithmic equation is equivalent to $3^{-4} = \dfrac{1}{81}$?

$$3^{-4} = \frac{1}{81} \quad \text{is equivalent to} \quad \log_3 \frac{1}{81} = -4. \quad \square$$

Example.

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8.$$

$$\log_5 \frac{1}{25} = -2 \quad \text{because} \quad 5^{-2} = \frac{1}{25}.$$

$$\log_2 \frac{1}{16} = -4 \quad \text{because} \quad 2^{-4} = \frac{1}{16}.$$

$$\log_{32} 2 = \frac{1}{5} \quad \text{because} \quad 32^{1/5} = 2. \quad \square$$

Remember that e is the number whose decimal value is $2.718281828459045\ldots$. According to the definition above,

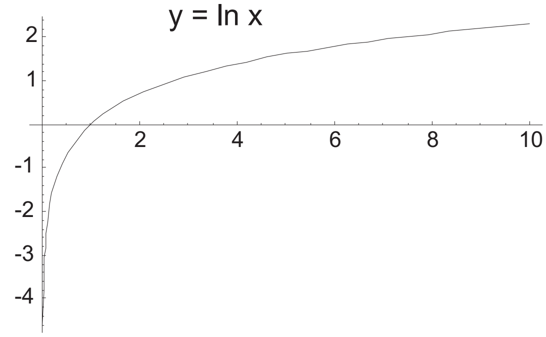
$$y = \log_e x \quad \text{means} \quad e^y = x.$$

Logs to the base e are so important that they have a special name; they are called **natural logarithms**. They also have a special symbol: Write lnx in place of $\log_e x$.

On most calculators, lnx and e^x are on the same button. For example, you can use your calculator to verify that

$$\ln 2 \approx 0.69315 \qquad \text{and} \qquad e^3 \approx 20.08554.$$

The graph of the natural logarithm is shown below:



$\ln x$ and e^x are **inverse functions** — if you do one, then the other, you get what you started with. In symbols,

$$\ln e^a = a \quad \text{for all} \quad a,$$

$$e^{\ln b} = b \quad \text{for} \quad b > 0.$$

These relationships are often useful for solving equations involving e^x or $\ln x$.

In addition, $\ln x$ satisfies the usual properties of logarithms.

Properties of the natural logarithm.

(a) $\ln 1 = 0$. (This makes sense, since $e^0 = 1$.)

(b) $\ln(ab) = \ln a + \ln b$.

(c) $\ln \dfrac{a}{b} = \ln a - \ln b$.

(d) $\ln a^p = p \ln a$.

Example. Solve $4^x = 5$.

To get something out of an exponent, take logs:

$$\ln 4^x = \ln 5, \quad x \ln 4 = \ln 5, \quad x = \frac{\ln 5}{\ln 4} \approx 1.16096. \quad \square$$

Example. Solve $\ln x + \ln(x - 5) = \ln 6$.

To get something out of a log, use e^x:

$$\ln x(x-5) = \ln 6, \quad e^{\ln x(x-5)} = e^{\ln 6}, \quad x(x-5) = 6, \quad x^2 - 5x = 6,$$

$$x^2 - 5x - 6 = 0, \quad (x-6)(x+1) = 0, \quad x = 6 \qquad \text{or} \qquad x = -1.$$

Check the possible solutions by plugging back in:

$$\ln 6 + \ln(6-5) = \ln 6 + \ln 1 = \ln 6. \quad \text{(Checks!)}$$

$$\ln(-1) + \ln(-1-5) \quad \text{is undefined.}$$

Therefore, the only solution is $x = 6$. $\square$

Example. \$1000 is to be invested at 4% annual interest, compounded quarterly. For how many years must the investment be held to accrue to at least \$10000?

Let n be the number of years required. Then

$$10000 = 1000\left(1 + \frac{0.04}{4}\right)^{4n}, \quad 10 = \left(1 + \frac{0.04}{4}\right)^{4n}, \quad \ln 10 = \ln(1.01)^{4n},$$

$$\ln 10 = 4n \ln 1.01, \quad n = \frac{\ln 10}{4 \ln 1.01} \approx 57.85197 \text{ years.} \quad \square$$

Example. Write $\log_3 \sqrt[5]{\dfrac{x^6y^4}{z^3}}$ in terms of $\log_3 x$, $\log_3 y$, and $\log_3 z$.

$$\log_3 \sqrt[5]{\frac{x^6y^4}{z^3}} = \frac{1}{5}\left(6\log_3 x + 4\log_3 y - 3\log_3 z\right). \quad \square$$

Example. Write $4\ln x + 3\ln 2 - 6\ln y$ as a single log.

$$4\ln x + 3\ln 2 - 6\ln y = \ln\frac{8x^4}{y^6}. \quad \square$$

Example. Solve $2^{2x} = 3$.

To get something out of an exponent (the $2x$), take logs:

$$\ln 2^{2x} = \ln 3, \quad 2x\ln 2 = \ln 3.$$

Then

$$\begin{array}{rccc} & 2x\ln 2 & = & \ln 3 \\ / & 2\ln 2 & & 2\ln 2 \\ \hline & x & = & \dfrac{\ln 3}{2\ln 2} \end{array}$$

The solution is $x = \dfrac{\ln 3}{2\ln 2} \approx 0.79248$. $\square$

Example. Solve $5^{2x} = 5^{3x-2}$.

Take logs on both sides:

$$\ln 5^{2x} = \ln 5^{3x-2}, \quad 2x\ln 5 = (3x-2)\ln 5.$$

Then

$$\begin{array}{rccc} & 2x\ln 5 & = & (3x-2)\ln 5 \\ / & \ln 5 & & \ln 5 \\ \hline & 2x & = & 3x - 2 \end{array}$$

And so

$$\begin{array}{rcccccc} & 2x & = & 3x & - & 2 \\ - & 3x & & 3x & & \\ \hline & -x & = & & & -2 \\ \times & -1 & & & & -1 \\ \hline & x & = & & & 2 \end{array}$$

The solution is $x = 2$. $\square$

Example. How many years will it take \$5000 invested at 4.8% annual interest, compounded monthly, to accrue to \$10000?

Plug $P = 5000$, $A = 10000$, $r = 0.048$, $k = 12$ in the compound interest formula

$$A = P\left(1 + \frac{r}{k}\right)^{nk}.$$

I get

$$10000 = 5000\left(1 + \frac{0.048}{12}\right)^{12n}, \quad 10000 = 5000 \cdot 1.004^{12n}.$$

Then

$$\begin{array}{rcl} 10000 & = & 5000 \cdot 1.004^{12n} \\ / \quad 5000 & & 5000 \\ \hline 2 & = & 1.004^{12n} \end{array}$$

Take logs on both sides:

$$\ln 2 = \ln 1.004^{12n}, \quad \ln 2 = 12n \ln 1.004.$$

So

$$\begin{array}{rcl} \ln 2 & = & 12n \ln 1.004 \\ / \quad 12 \ln 1.004 & & 12 \ln 1.004 \\ \hline \dfrac{\ln 2}{12 \ln 1.004} & = & n \end{array}$$

Thus, $n = \dfrac{\ln 2}{12 \ln 1.004} \approx 14.46943$ years. □

Example. Solve $\log_x \dfrac{1}{81} = 4$.

$\log_x \dfrac{1}{81} = 4$ means that $x^4 = \dfrac{1}{81}$. Therefore,

$$x = \pm \sqrt[4]{\frac{1}{81}} = \pm \frac{1}{3}.$$

The base of a logarithm must be positive, so $x = \dfrac{1}{3}$. □

Example. Compute $\log_2 3$ on your calculator.

Your calculator can compute logs to the base 10 and natural logs, which are logs to the base e. To compute logs to other bases, use the conversion formula

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

You t ake c to be a base available on your calculator. For example, using natural logs,

$$\log_a b = \frac{\ln b}{\ln a}.$$

So in this case,

$$\log_2 3 = \frac{\ln 3}{\ln 2} \approx 1.58496. \quad □$$

Example. Solve $\ln x + \ln(x - 1) = \ln(x + 3)$.

$$\ln x + \ln(x-1) = \ln(x+3), \quad \ln x(x-1) = \ln(x+3), \quad e^{\ln x(x-1)} = e^{\ln(x+3)},$$

$$x(x-1) = x + 3, \quad x^2 - x = x + 3.$$

Then

$$\begin{array}{rrrrrrrrr} & x^2 & - & x & & = & x & + & 3 \\ - & & & x & 3 & & x & & 3 \\ \hline & x^2 & - & 2x & - & 3 & = & & 0 \end{array}$$

Factor and solve:

$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$x - 3 = 0 \qquad\qquad x + 1 = 0$$
$$x = 3 \qquad\qquad x = -1$$

Check: If $x = 3$,

$$\ln x + \ln(x-1) = \ln 3 + \ln 2 = \ln 6 = \ln(x+3).$$

But if $x = -1$, $\ln x$ is undefined.
Hence, the only solution is $x = 3$. □

Example. Solve for x: $2\ln x + \ln(x-4) = \ln 5 + \ln x$.

$$2\ln x + \ln(x-4) = \ln 5 + \ln x, \quad \ln x^2 + \ln(x-4) = \ln 5 + \ln x, \quad \ln x^2(x-4) = \ln 5x,$$

$$e^{(\ln x^2(x-4))} = e^{(\ln 5x)}, \quad x^2(x-4) = 5x, \quad x^3 - 4x^2 = 5x.$$

Then

$$\begin{array}{rrrrrrrr} & x^3 & - & 4x^2 & & & = & 5x \\ - & & & & & 5x & & 5x \\ \hline & x^3 & - & 4x^2 & - & 5x & = & 0 \end{array}$$

Factor and solve:

$$x^3 - 4x^2 - 5x = 0$$
$$x(x-5)(x+1) = 0$$
$$\swarrow \qquad \downarrow \qquad \searrow$$
$$x = 0 \qquad x - 5 = 0 \qquad x + 1 = 0$$
$$x = 5 \qquad x = -1$$

Check: $x = 0$ and $x = -1$ can't be substituted into the original equation, because you can't take the log of 0 or a negative number.
If $x = 5$,

$$2\ln x + \ln(x-4) = 2\ln 5 + \ln(5-4) = 2\ln 5, \quad \ln 5 + \ln x = \ln 5 + \ln 5 = 2\ln 5.$$

The only solution is $x = 5$. □

Example. Solve for x: $e^{2x} - 7e^x - 8 = 0$.

Since $e^{2x} = (e^x)^2$, the equation is

$$(e^x)^2 - 7e^x - 8 = 0.$$

Factor and solve:

$$(e^x)^2 - 7e^x - 8 = 0$$
$$(e^x - 8)(e^x + 1) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$e^x - 8 = 0 \qquad\qquad e^x + 1 = 0$$
$$e^x = 8 \qquad\qquad e^x = -1$$

$e^x = -1$ is impossible, because $e^x > 0$ for all x.
To solve $e^x = 8$, take logs of both sides:

$$e^x = 8, \quad \ln e^x = \ln 8, \quad x = \ln 8.$$

The only solution is $x = \ln 8$. $\square$

Example. How much must be invested at 3.6% annual interest, compounded monthly, to accrue to \$3000 after 2 years?

Using the compound interest formula,

$$3000 = P\left(1 + \frac{0.036}{12}\right)^{2 \cdot 12}, \quad 3000 = P\left(1.003^{24}\right).$$

Therefore,

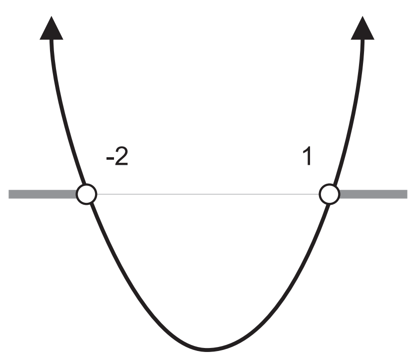
$$P = \frac{3000}{1.003^{24}} \approx 2791.89359. \quad \square$$

Example. If $\log_a x = 1.7$ and $\log_a y = 2.6$, compute $\log_a \dfrac{x^3}{\sqrt{y}}$.

$$\log_a \frac{x^3}{\sqrt{y}} = \log_a x^3 - \log_a \sqrt{y} = 3\log_a x - \frac{1}{2}\log_a y = 3 \cdot 1.7 - \frac{1}{2} \cdot 2.6 = 3.8. \quad \square$$

Example. Find the domain of $f(x) = \ln(x-1)(x+2)$.

You can't take the log of 0 or a negative number, so f is only defined if $(x-1)(x+2) > 0$.
$y = (x-1)(x+2)$ is a parabola opening upward with roots at $x = 1$ and $x = -2$.



The domain is $x < -2$ or $x > 1$. $\square$

Example. Solve for x: $(\ln x)^2 - 2\ln x = 35$.

$$\begin{array}{rrrrrr} & (\ln x)^2 & - & 2\ln x & & = & 35 \\ - & & & & 35 & & 35 \\ \hline & (\ln x)^2 & - & 2\ln x & - 35 & = & 0 \end{array}$$

Factor and solve:

$$(\ln x)^2 - 2\ln x - 35 = 0$$
$$(\ln x - 7)(\ln x + 5) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$\ln x - 7 = 0 \qquad\qquad \ln x + 5 = 0$$
$$\ln x = 7 \qquad\qquad \ln x = -5$$

Solve the two equations by exponentiating:

$$\ln x = 7, \quad e^{(\ln x)} = e^7, \quad x = e^7.$$

$$\ln x = -5, \quad e^{(\ln x)} = e^{-5}, \quad x = e^{-5}.$$

The solutions are $x = e^7$ and $x = e^{-5}$. □

Example. Solve for x: $4^{2x} = 6^{x+1}$.

$$4^{2x} = 6^{x+1}, \quad \ln 4^{2x} = \ln 6^{x+1}, \quad 2x\ln 4 = (x+1)\ln 6, \quad (2\ln 4)x = (\ln 6)x + \ln 6.$$

Then

$$\begin{array}{rrrcrcr} & (2\ln 4)x & & = & (\ln 6)x & + & \ln 6 \\ - & & (\ln 6)x & & (\ln 6)x & & \\ \hline & (2\ln 4)x & - \ (\ln 6)x & = & & & \ln 6 \end{array}$$

Then

$$(2\ln 4)x - (\ln 6)x = \ln 6, \quad (2\ln 4 - \ln 6)\,x = \ln 6, \quad x = \frac{\ln 6}{2\ln 4 - \ln 6} \approx 1.82678. \quad □$$
