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Equations Which Are Quadratic in Form

Sometimes an equation is not quadratic as is, but becomes quadratic if you make a **substitution**. Then you can solve the resulting quadratic, and get solutions to the original equation by using the substitution equation. Equations of this type are said to be **quadratic in form**.

Example. Solve $x^6 - 3x^3 - 4 = 0$. (Complex solutions are allowed.)

Write the equation as

$$(x^3)^2 - 3x^3 = 4 = 0.$$

The equation is actually quadratic; you can see this more clearly by substituting y for x^3:

$$y^2 - 3y - 4 = 0.$$

Factor and solve:

$$\begin{array}{c} y^2 - 3y - 4 = 0 \\ (y - 4)(y + 1) = 0 \end{array}$$

$$\begin{array}{ccc} & \swarrow \qquad \searrow & \\ y - 4 = 0 & & y + 1 = 0 \\ y = 4 & & y = -1 \end{array}$$

$y = 4$ gives $x^3 = 4$, or $x = \sqrt[3]{4}$.
$y = -1$ gives $x^3 = -1$, or $x = \sqrt[3]{-1} = -1$.
The solutions are $x = \sqrt[3]{4}$ and $x = -1$. □

Example. Solve $(x + 3)^2 - 4(x + 3) - 5 = 0$. (Complex solutions are allowed.)

Let $y = x + 3$. The equation becomes

$$y^2 - 4y - 5 = 0.$$

Factor and solve:

$$\begin{array}{c} y^2 - 4y - 5 = 0 \\ (y - 5)(y + 1) = 0 \end{array}$$

$$\begin{array}{ccc} & \swarrow \qquad \searrow & \\ y - 5 = 0 & & y + 1 = 0 \\ y = 5 & & y = -1 \end{array}$$

$y = 5$ gives $x + 3 = 5$, or $x = 2$.
$y = -1$ gives $x + 3 = -1$, or $x = -4$.
The solutions are $x = 2$ or $x = -4$. □

Example. Solve the equation $x^{-2} - 3x^{-1} - 4 = 0$. (Complex solutions are allowed.)

Write the equation as

$$(x^{-1})^2 - 3x^{-1} - 4 = 0.$$

Let $y = x^{-1}$. Then

$$y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0$$

$\swarrow$ $\searrow$

$$y = 4 \qquad y = -1$$
$$x^{-1} = 4 \qquad x^{-1} = -1$$
$$\frac{1}{x} = 4 \qquad \frac{1}{x} = -1$$
$$x = \frac{1}{4} \qquad x = -1 \quad \square$$

Example. Solve the equation $x^{2/3} - x^{1/3} - 2 = 0$. (Complex solutions are allowed.)

Write the equation as

$$(x^{1/3})^2 - x^{1/3} - 2 = 0.$$

Let $y = x^{1/3}$. Then

$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$

$\swarrow$ $\searrow$

$$y = 2 \qquad y = -1$$
$$x^{1/3} = 4 \qquad x^{1/3} = -1$$
$$(x^{1/3})^3 = 4^3 \qquad (x^{1/3})^3 = (-1)^3$$
$$x = 64 \qquad x = -1 \quad \square$$

Example. Solve the equation $(2x+3)^2 + 2(2x+3) - 3 = 0$. (Complex solutions are allowed.)

Let $y = 2x + 3$. Then

$$y^2 + 2y - 3 = 0$$
$$(y-1)(y+3) = 0$$

$\swarrow$ $\searrow$

$$y = 1 \qquad y = -3$$
$$2x + 3 = 1 \qquad 2x + 3 = -3$$
$$2x = -2 \qquad 2x = -6$$
$$x = -1 \qquad x = -3 \quad \square$$

Example. Solve $x^4 - 2x^2 - 3 = 0$. (Complex solutions are allowed.)

Write the equation as

$$(x^2)^2 - 2x^2 - 3 = 0.$$

Let $y = x^2$. The equation becomes

$$y^2 - 2y - 3 = 0.$$

Factor and solve:

$$y^2 - 2y - 3 = 0$$
$$(y-3)(y+1) = 0$$

$\swarrow$ $\searrow$

$$y - 3 = 0 \qquad y + 1 = 0$$
$$y = 3 \qquad y = -1$$

$y = 3$ gives $x^2 = 3$, or $x = \pm\sqrt{3}$.
$y = -1$ gives $x = -1$, or $x = \pm i$.
The solutions are $x = \pm\sqrt{3}$ and $x = \pm i$.
Note: If the problem had asked for only *real* solutions, then the only solutions are $x = \pm\sqrt{3}$. $\square$
