4-9-2008

Radical Equations

In this section, I'll discuss how you solve equations involving square roots of variable expressions.

The idea in solving such equations is to square both sides of the equation— sometimes several times — to eliminate the radicals. It's important to check the solutions when you're done, because it's possible for this procedure to produce bogus solutions.

Example. Solve $\sqrt{7x+2} = 3$.

Square both sides:

$$\begin{aligned} \sqrt{7x+2} &= 3 \\ (\sqrt{7x+2})^2 &= 3^2 \\ 7x+2 &= 9 \end{aligned}$$

Then

$$\begin{array}{rrrrrr} & 7x & + & 2 & = & 9 \\ - & & & 2 & & 2 \\ \hline & 7x & & & = & 7 \\ \div & 7 & & & & 7 \\ \hline & x & & & = & 1 \end{array}$$

Check: When $x = 1$,

$$\sqrt{7x+2} = \sqrt{7+2} = \sqrt{9} = 3.$$

The solution is $x = 1$. □

Example. Solve $\sqrt{2x+1} = -1$.

$\sqrt{2x+1}$ can't be negative, because $\sqrt{\cdot}$ denotes the *nonnegative* square root by definition. Therefore, the equation has no solutions. □

Example. Solve $x + 3 = \sqrt{x+5}$.

Square both sides:

$$\begin{aligned} x+3 &= \sqrt{x+5} \\ (x+3)^2 &= (\sqrt{x+5})^2 \\ x^2+6x+9 &= x+5 \end{aligned}$$

Note that in multiplying out $(x+3)^2$ I was careful to remember the middle term ("$6x$"). **Forgetting the middle term is one of the most common mistakes made in solving this kind of problem.**

Now I have

$$\begin{array}{rrrrrrrrrr} & x^2 & + & 6x & + & 9 & = & x & + & 5 \\ - & & & x & & & & x & & \\ \hline & x^2 & + & 5x & + & 9 & = & & & 5 \\ - & & & & & 5 & & & & 5 \\ \hline & x^2 & + & 5x & + & 4 & = & & & 0 \end{array}$$

Factor and solve:

$$x^2 + 5x + 4 = 0$$
$$(x+4)(x+1) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$x + 4 = 0 \qquad\qquad x + 1 = 0$$
$$x = -4 \qquad\qquad x = -1$$

$x = -1$ checks when substituted in the original equation. However, $x = -4$ gives

$$x + 3 = (-4) + 3 = -1 \quad \text{but} \quad \sqrt{x+5} = \sqrt{-4+5} = 1.$$

So the only solution is $x = -1$. □

Example. Solve $\sqrt{x+7} = 1 + \sqrt{x+2}$.

Square both sides and multiply out:

$$\begin{aligned}(\sqrt{x+7})^2 &= (1+\sqrt{x+2})^2 \\ x + 7 &= 1 + 2\sqrt{x+2} + (x+2) \\ x + 7 &= x + 3 + 2\sqrt{x+2}\end{aligned}$$

Note that in the second step I was careful to remember the middle term in computing $(1+\sqrt{x+2})^2$.

Before squaring again, I want to isolate the square root. Otherwise, I'll just create more square root terms and I won't make any progress.

So

$$\begin{array}{rccccccccc} & x & + & 7 & = & x & + & 3 & + & 2\sqrt{x+2} \\ - & x & & 3 & & x & & 3 & & \\ \hline & & & 4 & = & & & & & 2\sqrt{x+2}\end{array}$$

Now square both sides and multiply out:

$$\begin{aligned}4^2 &= (2\sqrt{x+2})^2 \\ 16 &= 4(x+2) \\ 16 &= 4x + 8\end{aligned}$$

Then

$$\begin{array}{rcccccc} & 16 & = & 4x & + & 8 \\ - & 8 & & & & 8 \\ \hline & 8 & = & 4x & & \\ / & 4 & & 4 & & \\ \hline & 2 & = & x & & \end{array}$$

Check: If $x = 2$,

$$\sqrt{x+7} = \sqrt{9} = 3, \quad \text{while} \quad 1 + \sqrt{x+2} = 1 + \sqrt{4} = 3.$$

The solution is $x = 2$. □

Example. Solve $\sqrt{3x+1} = \sqrt{x+4} + 1$.

Square both sides:

$$\begin{aligned}\sqrt{3x+1} &= \sqrt{x+4} + 1 \\ (\sqrt{3x+1})^2 &= (\sqrt{x+4}+1)^2 \\ 3x + 1 &= x + 4 + 2\sqrt{x+4} + 1 \\ 3x + 1 &= x + 5 + 2\sqrt{x+4}\end{aligned}$$

Isolate the radical:

$$\begin{array}{rrrrrrrrrr} & 3x & + & 1 & = & x & + & 5 & + & 2\sqrt{x+4} \\ - & x & & 5 & & x & & 5 & & \\ \hline & 2x & - & 4 & = & & & & & 2\sqrt{x+4} \\ \div & 2 & & 2 & & & & & & 2 \\ \hline & x & - & 2 & = & & & & & sqrtx+4 \end{array}$$

Now square both sides again:

$$(x-2)^2 = (\sqrt{x+4})^2$$
$$x^2 - 4x + 4 = x + 4$$

Solve for x:

$$\begin{array}{rrrrrrrrrr} & x^2 & - & 4x & + & 4 & = & x & + & 4 \\ - & & & x & & 4 & & x & & 4 \\ \hline & x^2 & - & 5x & & & = & 0 & & \\ & & x(x-5) & & & & = & 0 & & \end{array}$$

$$\swarrow \qquad \searrow$$

$$x = 0 \qquad\qquad x = 5$$

Check: When $x = 5$,

$$\sqrt{3x+1} = \sqrt{16} = 4 \quad \text{and} \quad \sqrt{x+4} + 1 = \sqrt{9} + 1 = 4.$$

When $x = 0$,

$$\sqrt{3x+1} = \sqrt{1} = 1 \quad \text{and} \quad \sqrt{x+4} + 1 = \sqrt{4} + 1 = 3.$$

$x = 0$ is not a solution.
The only solution is $x = 5$. □

Example. Solve $\sqrt{4x} - \sqrt{x-3} = \sqrt{x+5}$.

Square both sides:

$$\sqrt{4x} - \sqrt{x-3} = \sqrt{x+5}$$
$$(\sqrt{4x} - \sqrt{x-3})^2 = (\sqrt{x+5})^2$$
$$4x - 2\sqrt{4x}\sqrt{x-3} + (x-3) = x + 5$$

Isolate the radical:

$$5x - 3 - 2\sqrt{4x}\sqrt{x-3} = x + 5$$
$$4x - 8 = 2\sqrt{4x}\sqrt{x-3}$$
$$2x - 4 = \sqrt{4x}\sqrt{x-3}$$

Square both sides and solve:

$$(2x-4)^2 = (\sqrt{4x}\sqrt{x-3})^2$$
$$4x^2 - 16x + 16 = 4x(x-3)$$
$$4x^2 - 16x + 16 = 4x^2 - 12x$$
$$16 = 4x$$
$$x = 4$$

Check: When $x = 4$,

$$\sqrt{4x} - \sqrt{x-3} = \sqrt{16} - \sqrt{1} = 3 \quad \text{and} \quad \sqrt{x+5} = \sqrt{9} = 3.$$

The solution is $x = 4$. □
