Word Problems Involving Systems of Linear Equations

Many word problems will give rise to systems of equations — that is, a pair of equations like this:

$$\begin{aligned} 2x + 3y &= 10 \\ x - 6y &= 5 \end{aligned}$$

You can solve a system of equations in various ways. In many of the examples below, I'll use the **whole equation approach**. To review how this works, in the system above, I could multiply the first equation by 2 to get the y-numbers to match, then add the resulting equations:

$$\begin{aligned} 4x + 6y &= 20 \\ x - 6y &= 5 \\ \hline 5x &= 25 \\ x &= 5 \end{aligned}$$

If I plug $x = 5$ into $2x + 3y = 10$, I can solve for y:

$$\begin{aligned} 10 + 3y &= 10 \\ 3y &= 0 \\ y &= 0 \end{aligned}$$

In some cases, the whole equation method isn't necessary, because you can just do a substitution. You'll see this happen in a few of the examples.

The first few problems will involve items (coins, stamps, tickets) with different prices. If I have 6 tickets which cost \$15 each, the total cost is

$$6 \text{ tickets} \cdot 15 \, \frac{\text{dollars}}{\text{ticket}} = 90 \text{ dollars}.$$

If I have 8 dimes, the total value is

$$8 \text{ dimes} \cdot 10 \, \frac{\text{cents}}{\text{dime}} = 80 \text{ cents}.$$

This is common sense, and is probably familiar to you from your experience with coins and buying things. But notice that these examples tell me what the general equation should be: The number of items times the cost (or value) per item gives the total cost (or value). This is where I get the headings on the tables below.

You'll see that the same idea is used to set up the tables for all of these examples: Figure out what you'd do in a particular case, and the equation will say how to do this in general.

Example. Calvin has \$8.80 in pennies and nickels. If there are twice as many nickels as pennies, how many pennies does Calvin have? How many nickels?

In this kind of problem, it's good to do everything in cents to avoid having to work with decimals. So Calvin has 880 cents total.

Let p be the number of pennies. There are twice as many nickels as pennies, so there are $2p$ nickels. I'll arrange the information in a table.

	number of coins	$\cdot$	cents per coin	$=$	total value
pennies	p	$\cdot$	1	$=$	p
nickels	$2p$	$\cdot$	5	$=$	$10p$
total					880

Be sure you understand *why* the equations in the pennies and nickels rows are the way they are: The number of coins times the value per coin is the total value. If the words seem too abstract to grasp, try some examples:

If you have 3 nickels, they're worth $3 \cdot 5 = 15$ cents.

If you have 4 nickels, they're worth $4 \cdot 5 = 20$ cents.

If you have 5 nickels, they're worth $5 \cdot 5 = 25$ cents.

So if you have $2p$ nickels, they're worth $2p \cdot 5 = 10p$ cents.

The total value of the coins (880) is the value of the pennies **plus** the value of the nickels. So I add the first two numbers in the last column, then solve the resulting equation for p:

$$\begin{aligned} p + 10p &= 880 \\ 11p &= 880 \\ \frac{1}{11} \cdot 11p &= \frac{1}{11} \cdot 880 \\ p &= 80 \end{aligned}$$

Calvin has 80 pennies.
Therefore, he has $2p = 2 \cdot 80 = 160$ nickels. □

Tables for problems. I'll often arrange the equations for word problems in a **table**, as I did above. Roughly:

1. The *number* of things will go in the first column. This might be the number of tickets, the time it takes to make a trip, the amount of money invested in an account, and so on.

2. The *value per item* or *rate* will go in the second column. This might be the price per ticket, the speed of a plane, the interest rate (in percent) earned by an investment, and so on.

3. The *total value* or *total amount* will go in the third column. This might be the total cost of a number of tickets, the distance travelled by a car or a plane, the total interest earned by an investment, and so on.

With this arrangement:

$$\text{(first column)} \cdot \text{(second column)} = \text{(third column)}.$$

There are many correct ways of doing math problems, and you don't have to use tables to do these problems. But they are convenient for organizing information — and they give you a pattern to get started with problems of a given kind (e.g. interest problems, or time-speed-distance problems).

In some cases, you *add* the numbers in some of the columns in a table. In other cases, you set two of the numbers in a column equal, or subtract one number from another. There is no general rule for telling which of these things to do: You have to think about what the problem is telling you.

Example. A total of 78 seats for a concert are sold, producing a total revenue of \$483. If seats cost either \$2.50 or \$10.50, how many \$2.50 seats and how many \$10.50 seats were sold?

Suppose x of the \$2.50 seats and y of the \$10.50 seats were sold.

	seats	$\cdot$	price per seat	$=$	total revenue
\$2.50	x	$\cdot$	2.5	$=$	$2.5x$
\$10.50	y	$\cdot$	10.5	$=$	$10.5y$
total	78				483

The first and third columns give the equations

$$x + y = 78 \quad \text{and} \quad 2.5x + 10.5y = 483.$$

Multiply the second equation by 10 to clear decimals:

$$x + y = 78 \quad \text{and} \quad 25x + 105y = 4830.$$

Solve the equations by multiplying the first equation by 25 and subtracting it from the second:

$$\begin{array}{rrrcr} & 25x & + & 105y & = & 4830 \\ & 25x & + & 25y & = & 1950 \\ - & \hline & & & 80y & = & 2880 \\ & & & y & = & 36 \end{array}$$

Then $x + 36 = 78$, so $x = 42$. Thus, 42 of the \$2.50 seats and 36 of the \$10.50 seats were sold. □

Example. Tickets to a concert cost either \$12 or \$15. A total of 300 tickets are sold, and the total receipts were \$4140. How many of each kind of ticket were sold?

	Number of tickets	$\cdot$	Price per ticket	=	Value
\$12 tickets	x	$\cdot$	12	=	$12x$
\$15 tickets	y	$\cdot$	15	=	$15y$
Total	300	$\cdot$		=	4140

The first and third columns give the equations

$$\begin{aligned} x + y &= 300 \\ 12x + 15y &= 4140 \end{aligned}$$

Multiply the first equation by 15 and subtract equations:

$$\begin{array}{r} 15x + 15y = 4500 \\ \underline{12x + 15y = 4140} \\ 3x = 360 \\ x = 120 \end{array}$$

Then

$$\begin{aligned} x + y &= 300 \\ 120 + y &= 300 \\ y &= 180 \end{aligned}$$

There were 120 tickets sold for \$12 each and 180 tickets sold for \$15 each. □

Example. An investor buys a total of 360 shares of two stocks. The price of one stock is \$35 per share, while the price of the other stock is \$45 per share. The investor spends a total of \$15000. How many shares of each stock did the investor buy?

Let x be the number of shares of the \$35 stock and let y be the number of shares of the \$45 stock.

	Shares	$\cdot$	Price per share	$=$	Total cost
\$35 stock	x	$\cdot$	35	$=$	$35x$
\$45 stock	y	$\cdot$	45	$=$	$45y$
Total	360				15000

The first and third columns give

$$x + y = 360 \quad \text{and} \quad 35x + 45y = 15000.$$

Multiply the first equation by 45, then subtract the second equation:

$$\begin{aligned} 45x + 45y &= 16200 \\ 35x + 45y &= 15000 \\ \hline 10x &= 1200 \\ x &= 120 \end{aligned}$$

Since $x + y = 360$, I have $y = 360 - 120 = 240$. The investor bought 120 shares of the \$35 stock and 240 shares of the \$45 stock. □

The next problem is more complicated than the others, since it involves solving a system of *three* equations with *three* variables. You'll see that I do it by substitution. If you take more advanced courses (such as **linear algebra**), you'll learn methods for solving systems like these which are like the whole equation method. They involve representing the equations using **matrices**.

Example. Phoebe has some 32-cent stamps, some 29-cent stamps, and some 3-cent stamps. The number of 29-cent stamps is 10 less than the number of 32-cent stamps, while the number of 3-cent stamps is 5 less than the number of 29-cent stamps. The total value of the stamps is \$9.45. How many of each stamp does she have?

I will do everything in cents.

I'll let x be the number of 32-cent stamps, let y be the number of 29-cent stamps, and let z be the number of 3-cent stamps. Here's the table.

	number of stamps	$\cdot$	value per stamp	$=$	total value
32-cent	x	$\cdot$	32	$=$	$32x$
29-cent	y	$\cdot$	29	$=$	$29y$
3-cent	z	$\cdot$	3	$=$	$3z$
total					945

The last column says

$$32x + 29y + 3z = 945.$$

The number of 29-cent stamps is 10 less than the number of 32-cent stamps, so

$$y = x - 10.$$

The number of 3-cent stamps is 5 less than the number of 29-cent stamps, so

$$z = y - 5.$$

I want to get everything in terms of one variable, so I have to pick a variable to use. Since the last two equations both involve y, I'll do everything in terms of y.

z is already solved for in terms of y, since $z = y - 5$. I'll solve for x in terms of y:

$$\begin{aligned} y &= x - 10 \\ y + 10 &= x - 10 + 10 \\ y + 10 &= x \end{aligned}$$

Plug $x = y + 10$ and $z = y - 5$ into $32x + 29y + 3z = 945$ and solve for y:

$$\begin{aligned} 32(y + 10) + 29y + 3(y - 5) &= 945 \\ 32y + 320 + 29y + 3y - 15 &= 945 \\ 64y + 305 &= 945 \\ 64y + 305 - 305 &= 945 - 305 \\ 64y &= 640 \\ \frac{1}{64} \cdot 64y &= \frac{1}{64} \cdot 640 \\ y &= 10 \end{aligned}$$

Then

$$x = y + 10 = 10 + 10 = 20 \quad \text{and} \quad z = y - 5 = 10 - 5 = 5.$$

Phoebe has 20 32-cent stamps, 10 29-cent stamps, and 5 3-cent stamps. □

The next problem is about numbers. The setup will give two equations, but I don't need to solve them using the whole equation approach as I did in other problems. Since one variable is already solved for in the second equation, I can just substitute for it in the first equation.

Example. The sum of two numbers is 90. The larger number is 14 more than 3 times the smaller number. Find the numbers.

Let L be the larger number and let S be the smaller number. The sum is 90:

$$L + S = 90.$$

The larger number is 14 more than 3 times the smaller number:

$$L = 14 + 3S.$$

Plug $L = 14 + 3S$ into the first equation and solve:

$$\begin{aligned} L + S &= 90 \\ (14 + 3S) + S &= 90 \\ 14 + 4S &= 90 \\ 4S &= 76 \\ S &= 19 \end{aligned}$$

Then $L = 14 + 3S = 14 + 3 \cdot 19 = 71$. The numbers are 19 and 71. □

The next set of examples involve **simple interest**. Here's how it works. Suppose you invest \$600 (the **principal**) in an account which pays 4% simple interest. At the end of one interest period, the interest you earn is

$$600 \cdot 0.04 = 24.$$

You now have $600 + 24 = 624$ dollars in your account.

Notice that you multiply the amount invested (the principal) by the interest rate (in percent) to get the amount of interest earned.

By the way — How does "percent" fit the pattern of the earlier problems, where I had things like "dollars per ticket" or "cents per nickel"? In fact, "percent" is short for "per centum", and *centum* is the Latin word for a hundred. So "4 percent" means "4 per 100". Since "per" translates to division, I get $4\% = \dfrac{4}{100} = 0.04$, as you probably know from earlier math courses.

Example. \$6000 is divided between two accounts, one paying 4% interest and the other paying 3% interest. At the end of one interest period, the interest earned by the 4% account exceeds the interest earned by the 3% account by \$65. How much was invested in each account?

	Amount	$\cdot$	Interest rate	$=$	Interest
3% account	x	$\cdot$	0.03	$=$	$0.03x$
4% account	y	$\cdot$	0.04	$=$	$0.04y$
	6000				

I have

$$\begin{aligned} x + y &= 6000 \\ 0.03x + 65 &= 0.04y \end{aligned}$$

Rewrite the equations:

$$\begin{aligned} x + y &= 6000 \\ 0.03x - 0.04y &= -65 \end{aligned}$$

Clear the decimals by multiplying the second equation by 100:

$$\begin{aligned} x + y &= 6000 \\ 3x - 4y &= -6500 \end{aligned}$$

Multiply the first equation by 3 and subtract equations to solve for y:

$$\begin{aligned} 3x + 3y &= 18000 \\ 3x - 4y &= -6500 \\ \hline 7y &= 24500 \\ y &= 3500 \end{aligned}$$

Then

$$\begin{aligned} x + y &= 6000 \\ x + 3500 &= 6000 \\ x &= 2500 \end{aligned}$$

\$2500 was invested at 3% and \$3500 was invested at 4%. □

Example. Bonzo invests some money at 2% interest. He also invests \$1700 more than twice that amount at 4% interest. At the end of one interest period, the total interest earned was \$278. How much was invested at each rate?

	Amount	$\cdot$	Interest rate	$=$	Interest
2% account	x	$\cdot$	0.02	$=$	$0.02x$
4% account	$1700 + 2x$	$\cdot$	0.04	$=$	$0.04(1700 + 2x)$
					278

The last column gives an equation which can be solved for x:

$$\begin{aligned} 0.02x + 0.04(1700 + 2x) &= 278 \\ 2x + 4(1700 + 2x) &= 27800 \\ 2x + 6800 + 8x &= 27800 \\ 10x + 6800 &= 27800 \\ 10x &= 21000 \\ x &= 2100 \end{aligned}$$

Then $1700 + 2 \cdot 2100 = 5900$, so \$2100 was invested at 2% and \$5900 was invested at 4%. □

There are various kinds of **mixture problems**. The first few involve mixtures of different things which cost different amounts per pound. For instance, if you have 4 pounds of candy which costs \$2 per pound, the total cost of the candy is

$$4 \text{ pounds} \cdot 2\, \frac{\text{dollars}}{\text{pound}} = 8 \text{ dollars}.$$

In other words, the number of pounds times the price per pound is the total cost.

Example. Calvin mixes candy that sells for \$2.00 per pound with candy that costs \$3.60 per pound to make 50 pounds of candy selling for \$2.16 per pound. How many pounds of each kind of candy did he use in the mix?

	Pounds	$\cdot$	Dollars per pound	$=$	Dollars
\$2 candy	x	$\cdot$	2	$=$	$2x$
\$3.60 candy	y	$\cdot$	3.6	$=$	$3.6y$
Mixture	50	$\cdot$	2.16	$=$	108

The first and third columns give two equations:

$$\begin{aligned} x + y &= 50 \\ 2x + 3.6y &= 108 \end{aligned}$$

Multiply the first equation by 2 and subtract equations:

$$\begin{array}{r} 2x + 2y = 100 \\ \underline{2x + 3.6y = 108} \\ -1.6y = -8 \\ y = 5 \end{array}$$

Then

$$\begin{aligned} x + y &= 50 \\ x + 5 &= 50 \\ x &= 45 \end{aligned}$$

He used 45 pounds of the \$2 candy and 5 pounds of the \$3.60 candy. □

Example. Phoebe wants to mix raisins worth \$1.60 per pounds with nuts worth \$2.45 per pound to make 17 pounds of a mixture worth \$2 per pound. How many pounds of raisins and how many pounds of nuts should she use?

Suppose she uses x pounds of raisins and y pounds of dried fruit.

	pounds	$\cdot$	price per pound	$=$	dollars
raisins	x	$\cdot$	1.6	$=$	$1.6x$
nuts	y	$\cdot$	2.45	$=$	$2.45y$
mixture	17	$\cdot$	2	$=$	34

The first and third columns give the equations

$$x + y = 17 \quad \text{and} \quad 1.6x + 2.45y = 34.$$

Multiply the second equation by 100 to clear the decimals. This gives

$$x + y = 17 \quad \text{and} \quad 160x + 245y = 3400.$$

Solve the equations by multiplying the first equation by 160 and subtracting it from the second:

$$\begin{array}{rcrcr} 160x & + & 245y & = & 3400 \\ 160x & + & 160y & = & 2720 \\ \hline - & & 85y & = & 680 \\ & & y & = & 8 \end{array}$$

Hence, $x + 8 = 17$ and $x = 9$. She needs 8 pounds of raisins and 9 pounds of nuts. □

Mixture problems do not always wind up with two equations to solve. Here's an example where the setup gives a single equation.

Example. How many pounds of coffee worth \$4 per pound must be mixed with 20 pounds of Elmer's Glue$^{\text{TM}}$ worth \$2 per pound to obtain a mixture worth \$3.60 per pound?

Let x be the number of pounds of coffee. Set up a table:

	pounds	$\cdot$	dollars per pound	$=$	dollars
coffee	x	$\cdot$	4	$=$	$4x$
Elmer's Glue	20	$\cdot$	2	$=$	40
mixture	$(x + 20)$	$\cdot$	3.6	$=$	$4x + 40$

The last line says $3.6(x+20) = 4x+40$. Solve for x:

$$\begin{aligned} 3.6x + 72 &= 4x + 40 \\ 3.6x + 32 &= 4x \\ 32 &= 0.4x \\ x &= 80 \end{aligned}$$

You will need 80 pounds of coffee. □

An **alloy** is a mixture of different kinds of metals. Suppose you have 50 pounds of an alloy which is 20% silver. Then the number of pounds of (pure) silver in the 50 pounds is

$$50 \text{ pounds} \cdot 20\% = 10 \text{ pounds}.$$

That is, the 50 pounds of alloy consists of 10 pounds of pure silver and $50 - 10 = 40$ pounds of other metals.

Notice that you multiply the number of pounds of alloy by the percentage of silver to get the number of pounds of (pure) silver.

Example. Phoebe mixes an alloy containing 14% silver with an alloy containing 24% silver to make 100 pounds of an alloy with 18% silver. How many pounds of each kind of alloy did she use?

	Pounds	·	Percent silver	=	Pounds silver
14% alloy	x	·	0.14	=	$0.14x$
24% alloy	y	·	0.24	=	$0.24y$
18% alloy	100	·	0.18	=	18

The first and third columns give two equations:

$$\begin{aligned} x + y &= 100 \\ 0.14x + 0.24y &= 18 \end{aligned}$$

Multiply the second equation by 100 to clear decimals:

$$\begin{aligned} x + y &= 100 \\ 14x + 24y &= 1800 \end{aligned}$$

Multiply the first equation by 14 and subtract equations:

$$\begin{aligned} 14x + 14y &= 1400 \\ 14x + 24y &= 1800 \\ \hline -10y &= -400 \\ y &= 40 \end{aligned}$$

Then

$$\begin{aligned} x + y &= 100 \\ x + 40 &= 100 \\ x &= 60 \end{aligned}$$

She used 60 pounds of the 14% alloy and 40 pounds of the 24% alloy. □

Other mixture problems involve **solutions**. For instance, a solution may be 20% acid, or 20% alcohol. What does this mean? Suppose you have 80 gallons of a solution which is 20% acid. Then the number of gallons of (pure) acid in the solution is

$$80 \text{ gallons} \cdot 20\% = 16 \text{ gallons}.$$

So you can think of the 80 gallons of solution as being made of 16 gallons of pure acid and $80 - 16 = 64$ gallons of pure water.

Notice that you multiply the gallons of solution by the percentage of acid to get the number of gallons of (pure) acid.

Example. How many gallons of each of a 60% acid solution and an 80% acid solution must be mixed to produce 50 gallons of a 74% acid solution?

	Gallons of solution	$\cdot$	Percent acid	$=$	Gallons of acid
60% solution	x	$\cdot$	0.6	$=$	$0.6x$
80% solution	y	$\cdot$	0.8	$=$	$0.8y$
74% solution	50	$\cdot$	0.74	$=$	37

The first and third columns give the equations

$$\begin{aligned} x + y &= 50 \\ 0.6x + 0.8y &= 37 \end{aligned}$$

Multiply the second equation by 10 to clear the decimals:

$$\begin{aligned} x + y &= 50 \\ 6x + 8y &= 370 \end{aligned}$$

Multiply the first equation by 6 and subtract equations:

$$\begin{aligned} 6x + 6y &= 300 \\ 6x + 8y &= 370 \\ \hline -2y &= -70 \\ y &= 35 \end{aligned}$$

Then

$$\begin{aligned} x + y &= 50 \\ x + 35 &= 50 \\ x &= 15 \end{aligned}$$

Use 15 gallons of the 60% solution and 35 gallons of the 80% solution. $\square$

Example. Amounts of a 35% alcohol solution and a 65% alcohol solution are to be mixed to produce 24 gallons of a 45% alcohol solution. How many gallons of the 35% alcohol solution and how many gallons of the 65% alcohol solution should be used?

Suppose x gallons of the 35% alcohol solution and y gallons of the 65% alcohol solution are used.

	gallons	$\cdot$	percent alcohol	$=$	gallons alcohol
35% solution	x	$\cdot$	0.35	$=$	$0.35x$
65% solution	y	$\cdot$	0.65	$=$	$0.65y$
45% solution	24	$\cdot$	0.45	$=$	10.8

The first and third columns give the equations

$$x + y = 24 \quad \text{and} \quad 0.35x + 0.65y = 10.8.$$

Multiply the second equation by 100 to clear decimals:

$$x + y = 24 \quad \text{and} \quad 35x + 65y = 1080.$$

Solve the equations by multiplying the first equation by 65 and subtracting the second:

$$\begin{array}{rrcrcr} & 65x & + & 65y & = & 1560 \\ & 35x & + & 65y & = & 1080 \\ \hline - & & & 30x & = & 480 \\ & & & x & = & 16 \end{array}$$

Then $16 + y = 24$, so $y = 8$. Thus, 16 gallons of the 35% solution and 8 gallons of the 65% solution must be used. □

Two angles are **complementary** if their sum is 90° — that is, if they add up to a right angle.
For example, 30° and 60° are complementary, because

$$30 + 60 = 90.$$

Example. Two angles are complementary, and the larger one is 14° more than 3 time the smaller one. Find the angles.

Let L be the larger angle, and let S be the smaller angle. The angles are complementary:

$$L + S = 90.$$

The larger one is 14° more than 3 time the smaller one:

$$L = 14 + 3S.$$

Plug $L = 14 + 3S$ into $L + S = 90$ and solve for S:

$$\begin{aligned} (14 + 3S) + S &= 90 \\ 14 + 4S &= 90 \\ 4S &= 76 \\ S &= 19 \end{aligned}$$

Then $L = 14 + 3 \cdot 19 = 71$. The smaller angle is 19° and the larger angle is 71°. □

Example. Two angles are complementary. One angle is 81° less than twice the other. Find the two angles.

The sum is 90°:

$$A + B = 90.$$

One angle is 81° less than twice the other:

$$A = 2B - 81.$$

Plug $A = 2B - 81$ into the first equation and solve:

$$\begin{aligned} A + B &= 90 \\ (2B - 81) + B &= 90 \\ 3B - 81 &= 90 \\ 3B &= 171 \\ B &= 57 \end{aligned}$$

Then $A = 2B - 81 = 2 \cdot 57 - 81 = 33$. The angles are 33° and 57°. □
