Antiderivatives

\( F(x) \) is an antiderivative of \( f(x) \) if

\[ \frac{dF(x)}{dx} = f(x). \]

Notation:

\[ \int f(x) \, dx = F(x) + C. \]

For example,

\[ \int x^3 \, dx = \frac{1}{4} x^4 + C, \quad \text{because} \quad \frac{d}{dx} \left( \frac{1}{4} x^4 \right) = x^3. \]

In fact, all of the following functions are antiderivatives of \( x^3 \), because they all differentiate to \( x^3 \):

\[ \frac{1}{4} x^4, \quad \frac{1}{4} x^4 + 1, \quad \frac{1}{4} x^4 - 13, \quad \frac{1}{4} x^4 + 157. \]

This is the reason for the “+C” in the notation: You can add any constant to the “basic” antiderivative \( \frac{1}{4} x^4 \) and come up with another antiderivative.

\( C \) is called the arbitrary constant.

Remark. (a) Antiderivatives are often referred to as indefinite integrals, and sometimes I’ll refer to \( \int f(x) \, dx \) as “the integral of \( f(x) \) with respect to \( x \)”. This terminology is actually a bit misleading, but it’s traditional, so I’ll often use it. There is another kind of “integral” — the definite integral — which is probably more deserving of the name.

(b) The notation “\( \int f(x) \, dx \)” will also be used for definite integrals. The integral sign \( \int \) is a stretched-out “S”, and comes from the fact that definite integrals are defined in terms of sums.

“\( \int ( ) \, dx \)” is a mathematical object called an operator, which roughly speaking is a function which takes functions as inputs and produces functions as outputs. Despite appearances, “\( dx \)” isn’t a separate thing; in fact, “\( \int ( ) \, dx \)” is the whole name of the antiderivative operator. It’s a weird name — it consists of three symbols (“\( \int \)”, “\( d \)”, and “\( x \)”), and has a space between the “\( \int \)” and the “\( dx \)” for the input function.

I’ll come back to this again when I discuss substitution, since at that point this can become a source of confusion.

Every differentiation formula has a corresponding antidifferentiation formula. This makes it easy to derive antidifferentiation rules from the rules for differentiation.

Theorem. (Power Rule) For \( n \neq -1 \),

\[ \int x^n \, dx = \frac{1}{n + 1} x^{n+1} + C. \]

Proof. This follows from the fact that

\[ \frac{d}{dx} \frac{1}{n + 1} x^{n+1} = x^n. \]

(Notice that the expression on the left is undefined if \( n = -1 \).) □
Example. Compute the following antiderivatives:

(a) \( \int x^{100} \, dx \).

(b) \( \int \sqrt{x} \, dx \).

(c) \( \int \frac{1}{x^5} \, dx \).

(d) \( \int \frac{1}{x^{5/3}} \, dx \).

(a) \[ \int x^{100} \, dx = \frac{1}{101} x^{101} + C. \]

(b) \[ \int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{2}{3} x^{3/2} + C. \]

(c) \[ \int \frac{1}{x^5} \, dx = \int x^{-5} \, dx = -\frac{1}{4} x^{-4} + C. \]

(d) \[ \int \frac{1}{x^{5/3}} \, dx = \int x^{-5/3} \, dx = -\frac{3}{2} x^{-2/3} + C. \]

Theorem.

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx. \]

\[ \int k \cdot f(x) \, dx = k \int f(x) \, dx, \quad \text{if } k \text{ is a constant.} \]

\[ \int k \, dx = kx + \int f(x) \, dx, \quad \text{if } k \text{ is a constant.} \]

Proof. I'll prove the first formula by way of example; see if you can prove the others.

Suppose that \( \frac{d}{dx} F(x) = f(x) \) and \( \frac{d}{dx} G(x) = g(x) \).

By definition, this means that

\[ \int f(x) \, dx = F(x) + C \quad \text{and} \quad \int g(x) \, dx = G(x) + C. \]

By the rule for the derivative of a sum,

\[ \frac{d}{dx} (F(x) + G(x)) = \frac{d}{dx} F(x) + \frac{d}{dx} G(x) = f(x) + g(x). \]

By definition, this means that

\[ \int (f(x) + g(x)) \, dx = F(x) + G(x) + C. \]
Example. Compute the following antiderivatives:

(a) \( \int 8x^{10} \, dx \).

(b) \( \int (3x^4 + 2x + 5) \, dx \).

(a) 
\[
\int 8x^{10} \, dx = 8 \int x^{10} \, dx = \frac{8}{11} x^{11} + C.
\]

(b) 
\[
\int (3x^4 + 2x + 5) \, dx = 3 \int x^4 \, dx + 2 \int x \, dx + 5 \int dx = \frac{3}{5} x^5 + x^2 + 5x + C. \quad \square
\]

Since the derivative of a product is not the product of the derivatives, you can’t expect that it would work that way for antiderivatives, either.

Example. Compute \( \int (x^2 - 1)(x^4 + 2) \, dx \).

To do this antiderivative, I don’t antidifferentiate \( x^2 - 1 \) and \( x^4 + 2 \) separately. Instead, I multiply out, then use the rules I discussed above.

\[
\int (x^2 - 1)(x^4 + 2) \, dx = \int (x^6 - x^4 + 2x^2 - 2) \, dx = \frac{1}{7} x^7 - \frac{1}{5} x^5 + \frac{2}{3} x^3 - 2x + C. \quad \square
\]

Likewise, the derivative of a quotient is not the quotient of the derivatives, and it doesn’t work that way for antiderivatives.

Example. Compute \( \int \frac{x^4 + 1}{x^2} \, dx \).

Don’t antidifferentiate \( x^4 + 1 \) and \( x^2 \) separately! Instead, divide the bottom into the top:

\[
\int \frac{x^4 + 1}{x^2} \, dx = \int (x^2 + x^{-2}) \, dx = \frac{1}{3} x^3 - x^{-1} + C. \quad \square
\]

Every differentiation rule gives an antiderivation rule. So

\[
\frac{d}{dx} \sin x = \cos x \quad \text{means that} \quad \int \cos x \, dx = \sin x + C.
\]

Example. Compute \( \int (5x^7 + 4 \cos x) \, dx \).
For example,
\[ \int (5x^7 + 4 \cos x) \, dx = \frac{5}{8} x^8 + 4 \sin x + C. \]

**Example.** \( \frac{dy}{dx} = \left( x^2 + \frac{1}{x^2} \right)^2 \) and \( y(1) = \frac{1}{5} \). Find \( y \).

To find \( y \), antidifferentiate \( \frac{dy}{dx} \):
\[
y = \int \frac{dy}{dx} \, dx = \int \left( x^2 + \frac{1}{x^2} \right)^2 \, dx = \int \left( x^4 + 2 + \frac{1}{x^4} \right) \, dx = \frac{1}{5} x^5 + 2x - 2 \frac{1}{3} x^3 + C.
\]

\( y(1) = \frac{1}{5} \):
\[
\frac{1}{5} = y(1) = \frac{1}{5} + 2 - 2 \frac{2}{3} + C
\]
\[
C = -\frac{4}{3}
\]

Therefore,
\[
y = \frac{1}{5} x^5 + 2x - 2 \frac{1}{3} x^3 - \frac{4}{3}.
\]

This process is a simple example of solving a differential equation with an initial condition.

**Example.** Suppose an object moves with constant acceleration \( a \). Its initial velocity is \( v_0 \), and its initial position is \( s_0 \). Find its position function \( s(t) \).

First, \( a(t) = v'(t) = \frac{dv}{dt} \) so
\[
v = \int a(t) \, dt = \int a \, dt = at + C.
\]
When \( t = 0 \), \( v = v_0 \), so
\[
v_0 = a \cdot 0 + C, \quad C = v_0.
\]
Therefore,
\[
v = at + v_0.
\]
Next, \( v(t) = s'(t) = \frac{ds}{dt} \) so
\[
s = \int v(t) \, dt = \int (at + v_0) \, dt = \frac{1}{2} at^2 + v_0 t + D.
\]
When \( t = 0 \), \( s = s_0 \):
\[
s_0 = \frac{1}{2} a \cdot 0 + v_0 \cdot 0 + D, \quad D = s_0.
\]
Therefore,
\[
s = \frac{1}{2} at^2 + v_0 t + s_0.
\]

For example, an object falling near the surface of the earth experiences a constant acceleration of \(-32\) feet per second per second (negative, since the object’s height \( s \) is decreasing). Its height at time \( t \) is
\[
s = -16t^2 + v_0 t + s_0.
\]

Here \( v_0 \) is its initial velocity and \( s_0 \) is the height from which it’s dropped.