Arc Length in $\mathbb{R}^n$

Let $f : [a, b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$. How would you find the length of the curve? One approach is to start by approximating the curve with segments.

Divide the base interval $[a, b]$ up into subintervals:

$$a_0 = a, \ a_1, \ a_2, \ldots a_{n-1}, \ a_n = b.$$  

This is called a **partition** of the subinterval. You may recall doing this to set up Riemann sums. If you plug the $a$’s into $f$, you get points on the curve which you can connect with segments. Here’s a pictures with 4 points and 3 segments:

$$f(t) \quad f(a_0) \quad f(a_1) \quad f(a_2) \quad f(a_3)$$

The sum of the segments lengths approximates the length of the curve.

**Definition.** A curve $f : [a, b] \to \mathbb{R}^n$ is **rectifiable** if the sums of the segment lengths have an **upper bound**: that is, there is a number $M$ such that for every partition of $[a, b]$, the sum of the segment lengths is less than $M$.

If a curve is rectifiable, the **length** of the curve is least upper bound of the numbers $M$ which bound the sums of segment lengths for all partitions.

If a curve has reasonable properties, we can compute the length using an integral.

**Theorem.** Suppose $f : [a, b] \to \mathbb{R}^n$ is a curve where $f$ is differentiable and $f’(t)$ is continuous. Then $f$ is rectifiable, and the length of $f$ is

$$\int_a^b \|f’(t)\| \, dt.$$  

While the proof is a little technical, we can see why this makes sense. $\|f’(t)\|$ is the **speed** of an object moving along the curve. In a small increment $\Delta t$ of time, the object moves a distance $\|f’(t)\| \Delta t$. If we let the time increments go to 0 and add up the distances by integrating, we get the distance travelled by the object, which is the length of the curve.

**Example.** Find the length of the curve

$$x = t^2 - t, \quad y = \sqrt{3}t^2, \quad z = \frac{2\sqrt{12}}{3} t^{3/2} + 1, \quad \text{for} \quad 0 \leq t \leq 1.$$  

$$\frac{dx}{dt} = 2t - 1, \quad \frac{dy}{dt} = 2\sqrt{3}t, \quad \frac{dz}{dt} = \sqrt{12}t^{1/2}. $$

$$\left( \frac{dx}{dt} \right)^2 = 4t^2 - 4t + 1, \quad \left( \frac{dy}{dt} \right)^2 = 12t^2, \quad \left( \frac{dz}{dt} \right)^2 = 12t.$$  

$$\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 = 16t^2 + 8t + 1 = (4t + 1)^2.$$
\[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 4t + 1. \]

The length is
\[ \int_0^1 (4t + 1) \, dt = [2t^2 + t]_0^1 = 3. \]

**Example.** Find the length of the curve

\[ x = e^{2t}, \quad y = e^{-2t}, \quad z = \sqrt{8t}, \quad \text{for} \quad 0 \leq t \leq 1. \]

\[ \frac{dx}{dt} = 2e^{2t}, \quad \frac{dy}{dt} = -2e^{-2t}, \quad \frac{dz}{dt} = \sqrt{8}. \]

\[ \left(\frac{dx}{dt}\right)^2 = 4e^{4t}, \quad \left(\frac{dy}{dt}\right)^2 = 4e^{-4t}, \quad \left(\frac{dz}{dt}\right)^2 = 8. \]

\[ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 4e^{4t} + 8 + 4e^{-4t} = 4(e^{4t} + 2 + e^{-4t}) = 4(e^{2t} + e^{-2t})^2. \]

\[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 2(e^{2t} + e^{-2t}). \]

The length is
\[ \int_0^1 2(e^{2t} + e^{-2t}) \, dt = [e^{2t} - e^{-2t}]_0^1 = e^2 - e^{-2} = 7.25372 \ldots. \]