Functions of Several Variables

A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is a **function of several variables** if \( n > 1 \) — that is, if there is more than one input variable.

For example a function \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) is a **parametrized surface** in \( \mathbb{R}^3 \). Here’s a picture of

\[
f(u, v) = ((u^2 - 1) \cos v, u^3, (u^2 - 1) \sin v).
\]

Or consider a function \( f : \mathbb{R}^2 \to \mathbb{R} \). Its **graph** is a surface in \( \mathbb{R}^3 \). Here’s a picture of the graph of

\[
f(x, y) = \frac{\sin(x + y)}{x^2 + y^2 + 1}:
\]

Functions of several variables occur in many real world situations — in fact, most measurable quantities depend on many factors or variables. For example, the temperature at a point in space may be a function of its coordinates \((x, y, z)\). The ideal gas law \( pV = nRT \) relates the pressure \( p \), the volume \( V \), and the temperature \( T \) of an ideal gas. I can regard any one of these three variables as a function of the other two; for example, writing \( p = \frac{nRT}{V} \) views \( p \) as a function of \( V \) and \( T \).

**Example.** A function \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by

\[
f(x, y, z) = (2x + y, x - z, x^2y + z).
\]

Evaluate \( f(1, 4, -1) \), \( f(2, 3, 0) \), and \( f(6, 1, 1) \).

You substitute values into a function of several variables in the obvious way. For instance, to evaluate \( f(1, 4, -1) \), I set \( x = 1 \), \( y = 4 \), and \( z = -1 \) in the formula for \( f \):

\[
f(1, 4, -1) = (2 \cdot 1 + 4, 1 - (-1), 1^2 \cdot 4 + (-1)) = (6, 2, 3).
\]

Likewise,

\[
f(2, 3, 0) = (7, 2, 12) \quad \text{and} \quad f(6, 1, 1) = (13, 5, 37). \]

\[ \Box \]
Definition. For a function \( f : \mathbb{R}^n \to \mathbb{R}^m \): 

(a) The **domain** is the set of all points in \( \mathbb{R}^n \) where \( f \) is defined.

(b) The **image** (or the **range**) is the set of all outputs of \( f \) in \( \mathbb{R}^m \).

Remark. I’m following the usual convention: For functions \( \mathbb{R}^n \to \mathbb{R}^m \) to refer to the set of points in \( \mathbb{R}^n \) where the function is defined as the **domain** of the function. Thus, for the function \( f(x, y) = \frac{1}{x-y} \), the “domain” is the set of points \((x, y)\) such that \( x - y \neq 0 \), i.e. \( x \neq y \).

In more advanced courses, a more precise definition of a function requires that the “domain” be included as part of the function’s definition. In that context, what we’re calling the “domain” is referred to as the **natural domain** (the “biggest possible set” where the function is defined).

Example. A function of 2 variables is defined by

\[
f(x, y) = \frac{x^2 + y^2 + 4}{y^2 - 4x^2}.
\]

Describe the set of points \((x, y)\) for which \( f \) is undefined. What is the **domain** of \( f \)?

\[
f(x, y) = \frac{x^2 + y^2 + 4}{y^2 - 4x^2}
\]

is undefined when the denominator is 0:

\[
y^2 - 4x^2 = 0
\]

\[
(y - 2x)(y + 2x) = 0
\]

Thus, \( f \) is undefined for points \((x, y)\) on either of the lines \( y = 2x \) or \( y = -2x \).

The domain is the set of points \((x, y)\) which are *not* on \( y = 2x \) or \( y = -2x \) – i.e. the points such that \( y \neq \pm 2x \).

Example. A function of 2 variables is defined by

\[
f(x, y) = \ln(5x - y).
\]

Describe the set of points \((x, y)\) for which \( f \) is undefined. What is the **domain** of \( f \)?

The natural log function is undefined for inputs which are less than or equal to 0. So \( f(x, y) = \ln(5x - y) \) will be undefined if

\[
5x - y \leq 0
\]

\[
5x \leq y
\]
That is, \( f \) is undefined at points \((x, y)\) where \( y \geq 5x \). They are the points lying on or above the line \( y = 5x \):

![Diagram showing the region above the line \( y = 5x \).]

The domain is the set of points \((x, y)\) below the line \( y = 5x \).

**Example.** A function of 2 variables is defined by

\[
f(x, y) = \sqrt{1 - x^2 - y^2}.
\]

Describe the set of points \((x, y)\) for which \( f \) is undefined. What is the domain of \( f \)?

Since the square root of a negative number is undefined, \( f(x, y) = \sqrt{1 - x^2 - y^2} \) is undefined for

\[
1 - x^2 - y^2 < 0, \quad \text{or} \quad x^2 + y^2 > 1.
\]

That is, \( f \) is undefined at points \((x, y)\) lying outside the unit circle \( x^2 + y^2 = 1 \).

The domain is the set of points \((x, y)\) on or inside the unit circle \( x^2 + y^2 = 1 \).

**Example.** Describe the image (or the range) of the function \( z = f(x, y) = \tan^{-1}(x + y) \).

The image of a function is the set of outputs. What are the outputs of the inverse tangent function \( \tan^{-1}() \)?

For any \( x \) and \( y \),

\[
-\frac{\pi}{2} < \tan^{-1}(x + y) < \frac{\pi}{2}.
\]

I know that \( \tan^{-1}x \) attains every value between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), and if I set \( y = 0 \) I have \( \tan^{-1}(x + y) = \tan^{-1}x \).

Therefore, the image of \( f \) is \(-\frac{\pi}{2} < z < \frac{\pi}{2}\). 

A function of several variables can be pictured in many ways. For example, you can draw the graph of a function of 2 variables \( z = f(x, y) \). Because plotting points in 3 dimensions is tedious and difficult, you’d probably use software to draw the graph.

For functions \( f : \mathbb{R}^n \to \mathbb{R} \), you may also get a “picture” of the function by drawing its level sets. A level set for \( f \) is obtained by setting \( f = c \), where \( c \) is a constant. By using different values for \( c \), you get a picture of the “levels” of the function.
You may have seen **topographic maps**, where the level curves are referred to as **contour lines**. They represent lines along which the altitude ("$z$") is constant:

![Topographic Map](https://commons.wikimedia.org/wiki/File:Topographic_map.png)

**Example.** Sketch the graph of $z = y - x^2$, and some of the contour lines.

Here’s the graph of the function, produced by a computer:

![Graph](https://example.com/graph.png)

I can use a computer to sketch the level curves, but this example is simple enough that I’ll analyze it first. I get level curves by setting $z$ to specific numbers and graphing the curves I get.

<table>
<thead>
<tr>
<th>Value of $c$</th>
<th>Equation for $z = c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$y - x^2 = -2$ or $y = x^2 - 2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$y - x^2 = -1$ or $y = x^2 - 1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$y - x^2 = 0$ or $y = x^2$</td>
</tr>
<tr>
<td>$1$</td>
<td>$y - x^2 = 1$ or $y = x^2 + 1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$y - x^2 = 2$ or $y = x^2 + 2$</td>
</tr>
</tbody>
</table>

You can see the level curves are a family of parabolas.

![Level Curves](https://example.com/level_curves.png)
For a function of 3 variables \( w = f(x, y, z) \), setting \( w = c \) for various numbers \( c \) produces **level surfaces**. If you interpret \( w = f(x, y, z) \) as the temperature at a point \((x, y, z)\) in space, then a level surface \( w = c \) is the set of points in space where the temperature is \( c \).

**Example.** Describe some level surfaces for the function \( w = x - y^2 - z^2 \).

<table>
<thead>
<tr>
<th>( w )</th>
<th>Equation of level surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4)</td>
<td>(-4 = x - y^2 - z^2 ) or ( x = y^2 + z^2 + 4 )</td>
</tr>
<tr>
<td>(-2)</td>
<td>(-2 = x - y^2 - z^2 ) or ( x = y^2 + z^2 + 2 )</td>
</tr>
<tr>
<td>(0)</td>
<td>(0 = x - y^2 - z^2 ) or ( x = y^2 + z^2 )</td>
</tr>
<tr>
<td>(2)</td>
<td>(2 = x - y^2 - z^2 ) or ( x = y^2 + z^2 - 2 )</td>
</tr>
<tr>
<td>(4)</td>
<td>(4 = x - y^2 - z^2 ) or ( x = y^2 + z^2 - 4 )</td>
</tr>
</tbody>
</table>

The level surfaces are paraboloids opening along the \( x \)-axis.