The Fundamental Theorem of Calculus - Mathematica Demonstration

The Fundamental Theorem of Calculus says, roughly, that the following processes undo each other:

\{ \text{finding slopes of tangent lines} \} \quad \text{undo} \quad \{ \text{finding areas by rectangle sums} \}

The first process is differentiation, and the second process is (definite) integration. To say that the two undo each other means that if you start with a function, do one, then do the other, you get the function you started with.

In equation form, you can say

\[ \int_a^b f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F(x) \quad \text{is an antiderivative of} \quad f(x). \]

I’ll use the symbolic mathematics program Mathematica to give empirical evidence that it works.

Here is a Mathematica function which takes a function \( f : \mathbb{R} \to \mathbb{R} \) and an increment \( dx \) and returns the function \( x \mapsto \frac{f(x + dx) - f(x)}{dx} \):

\[ \text{DifferenceQuotient}[f, dx] := \text{Function}[\frac{f[# + dx] - f[#]}{dx}] \]

In ordinary math notation, this is \( \frac{f(x + dx) - f(x)}{dx} \). There’s no limit here; I’m working with a specific number \( dx \) and finding the slope of the secant line. It will be close to the slope of the tangent if \( dx \) is small.

For example, suppose I take \( f(x) = \sin(x^2) \):

\[ f[x_] := \text{Sin}[x^2] \]

I’ll let \( dqf \) denote the difference quotient function with an increment of \( dx = 0.01 \):

\[ dqf = \text{DifferenceQuotient}[f, 0.01] \]

In ordinary math notation, this is

\[ dqf(x) = \frac{\sin(x + 0.01)^2 - \sin x^2}{0.01}. \]

Here is \( dqf \) evaluated at \( x = 1.3 \):

\[ dqf[1.3] \]

\[ -0.344167 \]

Since \( f'(1.3) \approx -0.309196 \), this is not a bad approximation.

On the other hand, here is a function which approximates the area under a curve:

\[ \text{RiemannSum}[f, start, dx] := \text{Function}[\text{Sum}[f[\text{start} + i dx], \{i, 0, \text{Floor}[(\# - \text{start})/dx]\}], dx] \]

This function returns a function whose value is a rectangle sum approximation to

\[ \int_{\text{start}}^{x} f(x) \, dx. \]

(Each rectangle has width \( dx \).)
For example, using \( f(x) = \sin(x^2) \),

\[
\text{sumf} = \text{RiemannSum}[f, 0, 0.01]
\]

produces a function which gives a rectangle sum approximation to \( \int_0^x \sin(x^2) \, dx \).

In this case, the rectangles have width 0.01.

Now I can use \( \text{sumf} \) to approximate the area under the curve from 0 to \( \sqrt{\pi} \):

\[
\text{sumf}[\text{Sqrt}[\pi]]
\]

0.894835

This is pretty close to the “actual” answer, as found by a numerical integration routine:

\[
\text{NIntegrate}[f[x], \{x, 0, \text{Sqrt}[\pi]\}]
\]

0.894831

Here comes the punch line. I'll feed \( f(x) = \sin(x^2) \) into the difference quotient function, and then feed the output into the rectangle sum function:

\[
\text{stepone} = \text{DifferenceQuotient}[f, 0.1]
\]

\[
\text{steptwo} = \text{RiemannSum}[\text{stepone}, 0.01]
\]

Finally, I'll graph the function \( \text{steptwo} \). For comparison, I've plotted the graph of \( f(x) = \sin(x^2) \) on the right:

\[
\text{Plot}[\text{steptwo}[x], \{x, 0, 2\text{Sqrt}[\pi]\}]
\]

\[
\text{Plot}[\sin[x^2], \{x, 0, 2\text{Sqrt}[\pi]\}]
\]

You can see that the two graphs are essentially the same, except for the “steps” on the graph. These result from the fact that I didn’t take limits in defining my tangent line approximation or my rectangle sum approximation. But you could make the “steps” smaller by making \( dx \) smaller in the \text{DifferenceQuotient} and \text{RiemannSum} functions.

The results which I’ve demonstrated empirically are summarized in the **The Fundamental Theorem of Calculus.**

**Theorem.** (The Fundamental Theorem of Calculus (first version)) Suppose \( f \) is integrable on \( a \leq x \leq b \), and that \( F'(x) = f(x) \) for some differentiable function \( F \) defined on \( a \leq x \leq b \). Then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

The Fundamental Theorem of Calculus says that I can compute the definite integral of a function \( f \) by finding an antiderivative \( F \) of \( f \).