Inverse Trig Functions

If you restrict $f(x) = \sin x$ to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the function increases:

This implies that the function is one-to-one, and hence it has an inverse. The inverse is called the inverse sine or arcsine function, and is denoted $\arcsin x$ or $\sin^{-1}(x)$. Note that in the second case $\sin^{-1}(x)$ does not mean $\frac{1}{\sin x}$!

Note: “Arcsine” (and $\arcsin x$) are older terms, and there is similar terminology for the other inverse trig functions (so “arctangent” and $\arctan x$ for the inverse tangent function, and so on). I’ll use the inverse function terminology instead.

In word, $y = \sin^{-1} x$ is the angle whose sine is $x$. Another way of saying this is:

$y = \sin^{-1} x$ is the same as $\sin y = x$.

The fact that $\sin$ and $\sin^{-1}$ are inverse functions can be expressed by the following equations:

$\sin \sin^{-1} a = a \text{ for } -1 \leq a \leq 1,$

$\sin^{-1} \sin b = b \text{ for } -\frac{\pi}{2} \leq b \leq \frac{\pi}{2}.$

Since the restricted $\sin$ takes angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and produces numbers in the range $-1 \leq y \leq 1$, $\sin^{-1}$ takes numbers in the range $-1 \leq y \leq 1$ and produces angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
**Example.** Compute $\sin^{-1} \frac{1}{2}$ and $\sin^{-1}(-1)$.

$$
\sin^{-1} \frac{1}{2} = \frac{\pi}{6}, \text{ since } \sin \frac{\pi}{6} = \frac{1}{2}.
$$

$$
\sin^{-1}(-1) = -\frac{\pi}{2}, \text{ since } \sin \left(-\frac{\pi}{2}\right) = -1.
$$

Sine and arcsine are inverses, so they undo one another — but you have to be careful!

$$
\sin \left( \arcsin \frac{2}{5} \right) = \frac{2}{5}, \text{ but } \arcsin (\sin 2\pi) = 0, \text{ not } 2\pi.
$$

$\sin^{-1}(\text{stuff})$ can’t be $2\pi$, because $\sin^{-1}$ always returns an angle in the range $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$.

**Example.** Find $\tan \sin^{-1} \frac{5}{13}$.

First, let $\theta = \sin^{-1} \frac{5}{13}$. This means that $\sin \theta = \frac{5}{13}$. Now $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, so I get the following picture:

![Right Triangle](image)

I got the adjacent side using Pythagoras: $\sqrt{13^2 - 5^2} = 12$.

Using the triangle, I have

$$
\tan \sin^{-1} \frac{5}{13} = \tan \theta = \frac{5}{12}.
$$

You can find a derivative formula for $\sin^{-1}$ using implicit differentiation. Let $y = \sin^{-1} x$. This is equivalent to $x = \sin y$. Differentiate implicitly:

$$
x = \sin y\quad 1 = (\cos y)y'\quad y' = \frac{1}{\cos y}
$$

I’d like to express the result in terms of $x$. Here’s the right triangle that says $x = \sin y$:

![Right Triangle](image)
I found the other leg using Pythagoras. You can see that $\cos y = \sqrt{1-x^2}$. Hence, $y' = \frac{1}{\sqrt{1-x^2}}$. That is,

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

Every derivative formula gives rise to a corresponding antiderivative formula:

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C.$$

Before I do some calculus examples, I want to mention some of the other inverse trig functions. I’ll discuss the inverse cosine, inverse tangent, and inverse secant functions.

(a) You get the inverse cosine by inverting $\cos x$, restricted to $0 \leq x \leq \pi$.

(b) You get the inverse tangent by inverting $\tan x$, restricted to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(c) You get the inverse secant by inverting \( \sec x \), restricted to \( 0 < x < \frac{\pi}{2} \) together with \( \frac{\pi}{2} < x < \pi \).

As with \( \sin \) and \( \sin^{-1} \), the domains and ranges of these functions and their inverses are “swapped”:

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1} )</td>
<td>( -1 \leq x \leq 1 )</td>
<td>( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \cos^{-1} )</td>
<td>( -1 \leq x \leq 1 )</td>
<td>( 0 \leq x \leq \pi )</td>
</tr>
<tr>
<td>( \tan^{-1} )</td>
<td>( -\infty &lt; x &lt; \infty )</td>
<td>( -\frac{\pi}{2} &lt; x &lt; \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \sec^{-1} )</td>
<td>( x \leq -1, x \geq 1 )</td>
<td>( 0 \leq x &lt; \frac{\pi}{2}, \frac{\pi}{2} &lt; x \leq \pi )</td>
</tr>
</tbody>
</table>

**Example.** Compute \( \tan^{-1} 1 \) and \( \cos^{-1} \left( -\frac{1}{2} \right) \).

\[
\tan^{-1} 1 = \frac{\pi}{4}, \quad \text{since} \quad \tan \frac{\pi}{4} = 1.
\]

\[
\cos^{-1} \left( -\frac{1}{2} \right) = \frac{2\pi}{3}, \quad \text{since} \quad \cos \frac{2\pi}{3} = -\frac{1}{2}
\]

You can derive the derivative formulas for the other inverse trig functions using implicit differentiation, just as I did for the inverse sine function.

\[
\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}
\]

\[
\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}
\]

\[
\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}
\]

For example, I’ll derive the formula for \( \frac{d}{dx} \sec^{-1} x \).
The derivation starts out like the derivation for \( \frac{d}{dx} \sin^{-1} x \). Let \( y = \sec^{-1} x \), so \( y = x \). Differentiating implicitly, I get

\[
(\sec y \tan y) y' = 1
\]

\[
y' = \frac{1}{\sec y \tan y}
\]

There are two cases, depending on whether \( x \geq 1 \) or \( x \leq -1 \).

Suppose \( x \geq 1 \). Then \( y = \sec^{-1} x \) is in the interval \( 0 \leq y < \frac{\pi}{2} \), as illustrated in the first diagram above. You can see from the picture that

\[
\sec y = x \quad \text{and} \quad \tan y = \sqrt{x^2 - 1}.
\]

Therefore,

\[
y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}.
\]

\( x \geq 1 \), so \( x \) is positive, and \( x = |x| \). Therefore,

\[
y' = \frac{1}{x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.
\]

Now suppose that \( x \leq -1 \). Then \( y = \sec^{-1} x \) is in the interval \( \frac{\pi}{2} < y \leq \pi \), as illustrated in the second diagram above. Since \( x \) is negative, the hypotenuse must be \(-x\), since it must be positive and since \( \sec y = \frac{\text{(hypotenuse)}}{\text{(adjacent)}} \) must equal \( x \). In this case,

\[
\sec y = x \quad \text{and} \quad \tan y = -\sqrt{x^2 - 1}.
\]

Therefore,

\[
y' = \frac{1}{\sec y \tan y} = \frac{1}{-x\sqrt{x^2 - 1}}.
\]

\( x \leq -1 \), so \( x \) is negative, and \( -x = |x| \). Therefore,

\[
y' = \frac{1}{-x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.
\]

This proves that \( y' = \frac{1}{|x|\sqrt{x^2 - 1}} \) in all cases.

**Example.** Compute:
(a) \[ \frac{d}{dx} \left( \sin^{-1} \sqrt{x} + \sqrt{\sin^{-1} x} \right). \]

(b) \[ \frac{d}{dx} \tan^{-1} x. \]

(c) \[ \frac{d}{dx} \sec^{-1}(e^x). \]

(a) \[ \frac{d}{dx} \left( \sin^{-1} \sqrt{x} + \sqrt{\sin^{-1} x} \right) = \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2} \left( \sin^{-1} x \right)^{-1/2} \cdot \frac{1}{\sqrt{1 - x^2}}. \]

(b) \[ \frac{d}{dx} \frac{1}{\tan^{-1} x} = \left( -\frac{1}{(\tan^{-1} x)^2} \right) \left( \frac{1}{1 + x^2} \right). \]

(c) \[ \frac{d}{dx} \sec^{-1}(e^x) = \frac{e^x}{e^x \sqrt{e^{2x} - 1}} = \frac{1}{\sqrt{e^{2x} - 1}}. \]

I don’t need absolute values in the last example, because \( e^x \) is always positive.

Example. Prove the identity 
\[ \tan^{-1} w + \tan^{-1} \frac{1}{w} = \frac{\pi}{2}. \]

\[ \frac{d}{dx} \tan^{-1} \frac{1}{w} = -\frac{1}{w^2} \cdot \frac{1}{1 + \frac{1}{w^2}} = -\frac{1}{1 + w^2}. \]

Hence, 
\[ \frac{d}{dx} \left( \tan^{-1} w + \tan^{-1} \frac{1}{w} \right) = 0. \]

A function with zero derivative is constant, so 
\[ \tan^{-1} w + \tan^{-1} \frac{1}{w} = C, \quad \text{a constant.} \]

But when \( w = 1 \), 
\[ C = \tan^{-1} w + \tan^{-1} \frac{1}{w} = \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{2}. \]

Therefore, 
\[ \tan^{-1} w + \tan^{-1} \frac{1}{w} = \frac{\pi}{2}. \]

Here are the integration formulas for some of the inverse trig functions. I’m giving extended versions of the formulas — with “\( a^2 \)” replacing the “1” that you’d get if you just reversed the derivative formulas — in order to save you a little time in doing problems.

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \]
\[
\int \frac{1}{|x|\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + C
\]

For instance, here’s how to derive the extended \(\sin^{-1}\) integral formula from the formula \(\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C\) using substitution:

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \, du = \frac{1}{a} \frac{du}{\sqrt{1 - u}} = \sin^{-1} u + C = \frac{1}{a} \sin^{-1} \frac{x}{a} + C.
\]

\[
\left[ u = \frac{x}{a}, \quad du = \frac{dx}{a}, \quad dx = a \, du \right]
\]

Example. Compute \(\int \frac{dx}{4 + x^2}\) and \(\int \frac{1}{\sqrt{3 - x^2}} \, dx\).

Using the \(\tan^{-1}\) formula with \(a = 2\),

\[
\int \frac{dx}{4 + x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} + C.
\]

Using the \(\sin^{-1}\) formula with \(a = \sqrt{3}\),

\[
\int \frac{1}{\sqrt{3 - x^2}} \, dx = \sin^{-1} \frac{x}{\sqrt{3}} + C.
\]

Example. Compute \(\int \frac{dx}{1 + 4x^2}\).

\[
\int \frac{dx}{1 + 4x^2} = \int \frac{dx}{1 + (2x)^2} = \int \frac{1}{1 + u^2} \cdot \frac{du}{2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(2x) + C.
\]

\[
\left[ u = 2x, \quad du = 2 \, dx, \quad dx = \frac{du}{2} \right]
\]

Example. Compute \(\int \frac{x^4 \, dx}{1 + x^{10}}\).

\[
\int \frac{x^4 \, dx}{1 + x^{10}} = \int \frac{x^4 \, dx}{1 + (x^5)^2} = \int \frac{x^4 \, du}{1 + u^2} = \frac{1}{5} \int \frac{du}{1 + u^2} = \\
\left[ u = x^5, \quad du = 5x^4 \, dx, \quad dx = \frac{du}{5x^4} \right]
\]

\[
\frac{1}{5} \tan^{-1} u + C = \frac{1}{5} \tan^{-1}(x^5) + C.
\]

Example. Compute \(\int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx\).
\[
\int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx = \int \frac{e^x}{\sqrt{1 - u^2}} \cdot \frac{du}{e^x} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.
\]

\[
\left[ u = e^x, \quad du = e^x \, dx, \quad dx = \frac{du}{e^x} \right] \quad \square
\]

Example. Compute \( \int \frac{(\sec x)^2 \, dx}{\sqrt{1 - (\tan x)^2}} \).

\[
\int \frac{(\sec x)^2 \, dx}{\sqrt{1 - (\tan x)^2}} = \int \frac{(\sec x)^2}{\sqrt{1 - u^2}} \cdot \frac{du}{(\sec x)^2} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1} \tan x + C.
\]

\[
\left[ u = \tan x, \quad du = (\sec x)^2 \, dx, \quad dx = \frac{du}{(\sec x)^2} \right] \quad \square
\]