Left and Right-Hand Limits

In some cases, you let \( x \) approach the number \( a \) from the left or the right, rather than “both sides at once” as usual.

- \( \lim_{x \to a^+} f(x) \) means: Compute the limit of \( f(x) \) as \( x \) approaches \( a \) from the right.
- \( \lim_{x \to a^-} f(x) \) means: Compute the limit of \( f(x) \) as \( x \) approaches \( a \) from the left.

The left- and right-hand limits are the same if and only if the ordinary limit exists. In this case, the left-hand, right-hand, and ordinary limit are equal.

Example.

\[
\lim_{x \to 0^+} \frac{|\sin x|}{\sin x} = 1, \quad \text{but} \quad \lim_{x \to 0^-} \frac{|\sin x|}{\sin x} = -1.
\]

Look at the first limit more closely. \( x \) approaches 0 from the right. Numbers close to, but to the right of, 0 are small positive numbers: 0.01, for example. Small positive numbers make \( \sin x \) positive: \( \sin 0.01 \approx 0.01000 \), for example. If \( \sin x \) is positive, then \( |\sin x| = \sin x \), so

\[
\frac{|\sin x|}{\sin x} = \frac{\sin x}{\sin x} = 1.
\]

(Notice that you don’t let \( x \) equal 0, so \( \sin x \neq 0 \), and the cancellation is legal.) Therefore,

\[
\lim_{x \to 0^+} \frac{|\sin x|}{\sin x} = \lim_{x \to 0^+} 1 = 1.
\]

Here’s the picture:

Since the left- and right-hand limits are not the same, \( \lim_{x \to 0} \frac{|\sin x|}{\sin x} \) is undefined.

Example. Suppose

\[
f(x) = \begin{cases} 
2x + 1 & \text{if } x < 1 \\
5 & \text{if } x = 1 \\
7x^2 - 4 & \text{if } x > 1 
\end{cases}
\]
Compute $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$, and $\lim_{x \to 1} f(x)$.

To compute $\lim_{x \to 1^+} f(x)$, I use the part of the definition for $f$ which applies to $x > 1$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x + 1) = 3.$$ 

Likewise, to compute $\lim_{x \to 1^-} f(x)$, I use the part of the definition for $f$ which applies to $x < 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (7x^2 - 4) = 3.$$ 

Since the left and right-hand limits are equal, the two-sided limit is defined, and $\lim_{x \to 1} f(x) = 3$.

The fact that $f(1) = 5$ does not come into the problem. 

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**Example.** Consider the function $f(x)$ whose graph is depicted below:

Then

$$\lim_{x \to 1^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 3.$$ 

Since the left- and right-hand limits are not the same,

$$\lim_{x \to 1} f(x) \text{ is undefined.}$$

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**Example.** Consider the function $f(x)$ whose graph is depicted below:
Then
\[ \lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = 1. \]

Therefore,
\[ \lim_{x \to 0} f(x) = 1. \]

The value of \( f(0) \) does not affect the existence of the limit. In fact, suppose I change the function as follows:

Now \( f(0) \) is undefined, but
\[ \lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0^-} f(x) = 1, \quad \text{and} \quad \lim_{x \to 0} f(x) = 1. \]

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**Example.** Compute \( \lim_{x \to 1^+} \frac{x^2 - 2x - 3}{x - 1} \).

Plugging in gives \(-\frac{4}{0}\). The limit is *undefined*. But I can say more.

Try plugging in a number close to 1: When \( x = 1.001 \),
\[ \frac{x^2 - 2x - 3}{x - 1} \approx -4000. \]

It looks as though \( \frac{x^2 - 2x - 3}{x - 1} \) is getting *big and negative*. In fact,
\[ \lim_{x \to 1^+} \frac{x^2 - 2x - 3}{x - 1} = -\infty. \]

To why this is true, remember that \( x \) is approaching 1 *from the right*. This means that \( x - 1 \) will be small and positive. On the other hand, \( x^2 - 2x - 3 \to -4 \). Since the top is negative and the bottom is positive, the result must be *negative*.

As far as size goes, I have
\[ \frac{\text{nonzero number}}{\text{small number}} = \text{big number}. \]

Since the result should be *big and negative*, it is reasonable that it is \(-\infty\).
Another way to see this is to draw the graph near $x = 1$. As you move toward 1 from the right, the graph goes downward toward $-\infty$.

I noted earlier that if \( \frac{f(x)}{g(x)} \rightarrow \text{nonzero number} \), then the two-sided limit \( \lim_{x \to c} \frac{f(x)}{g(x)} \) is undefined. As the example above shows, the situation is different with one-sided limits.

If, in this situation, \( g(x) \) has the same sign for all \( x \)'s sufficiently close to \( c \) and greater than \( c \), then the right-hand limit \( \lim_{x \to c^+} \frac{f(x)}{g(x)} \) will be either $+\infty$ or $-\infty$. The specific sign depends on the signs of the top and the bottom of the fraction.

Likewise, if \( g(x) \) has the same sign for all \( x \)'s sufficiently close to \( c \) and less than \( c \), then the left-hand limit \( \lim_{x \to c^-} \frac{f(x)}{g(x)} \) will be either $+\infty$ or $-\infty$. Again, the specific sign depends on the signs of the top and the bottom of the fraction.

The “same-sign” condition will be satisfied, for example, if \( f \) and \( g \) are polynomials — that is, if \( \frac{f(x)}{g(x)} \) is a rational function. It will also be satisfied by functions like \( \frac{x - 3}{x^{1/3} - 2} \) as \( x \to 8 \).

**Example.** Compute \( \lim_{x \to -3^+} \frac{x + 1}{x + 3} \).

Plugging \( x = -3 \) in gives $\frac{-2}{0}$. Since \( \frac{x + 1}{x + 3} \) is a rational function, the right-hand limit \( \lim_{x \to -3^+} \frac{x + 1}{x + 3} \) is either $+\infty$ or $-\infty$; I have to determine which of the two it is. I’ll look at the top and the bottom separately.

As \( x \to -3^+ \), \( x + 1 \to -2 \).

As for the bottom, since \( x \) is approaching $-3$ from the right, I’m considering \( x \)'s greater than $-3$. Thus, \( x > -3 \), so \( x + 3 > 0 \) — \( x + 3 \) is positive.

Since \( x + 1 \) is approaching a negative number and \( x + 3 \) is approaching a positive number, the quotient is negative. Therefore,

\[
\lim_{x \to -3^+} \frac{x + 1}{x + 3} = -\infty.
\]

I can also see this if I take a number close to $-3$ but to the right of $-3$ — $x = -2.99$, for example — and plug it in:

\[
\frac{x + 1}{x + 3} = \frac{-2.99 + 1}{-2.99 + 3} = \frac{-1.99}{0.01} = -199.
\]
I got a large negative number, which suggests that the limit should be $-\infty$.
I could also see this by graphing the function, as in the previous example.
You might ask: “Which of these methods is the best?” I feel that for a first course in calculus, all three are acceptable.

However, while plugging in numbers and drawing graphs provide support for a conclusion, they don’t really provide a proof. Graphs can be deceiving. And when you plug in a number, how do you know that the number you chose is “typical”? The first method — reasoning about signs using inequalities — is much closer to a rigorous proof of the result.