The Remainder Term

If the Taylor series for a function $f(x)$ is truncated at the $n^{th}$ term, what is the difference between $f(x)$ and the value given by the $n^{th}$ Taylor polynomial? That is, what is the error involved in using the Taylor polynomial to approximate the function?

**Theorem.** Suppose you expand $f$ around $c$, and that $f$ is $(n+1)$-times continuously differentiable on an open interval containing $c$. If $x$ is another point in this interval, then from some $z$ in the open interval between $x$ and $c$,

$$p_n(x; c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k$$

is the $n^{th}$ degree Taylor polynomial at $c$. The other term on the right is called the Lagrange remainder term:

$$R_n(x; c) = \frac{f^{(n+1)}(z)}{(n+1)!} (x - c)^{n+1}.$$

The appearance of $z$, a point between $x$ and $c$, and the fact that it’s being plugged into a derivative suggest that there is a connection between this result and the Mean Value Theorem. In fact, for $n = 0$ the result says that there is a number $z$ between $c$ and $x$ such that

$$f(x) = f(c) + f'(z) \cdot (x - c).$$

This is the Mean Value Theorem.

On the one hand, this reflects the fact that Taylor’s theorem is proved using a generalization of the Mean Value Theorem. On the other hand, this shows that you can regard a Taylor expansion as an *extension* of the Mean Value Theorem.

**Example.** Compute the Remainder Term $R_3(x; 1)$ for $f(x) = \sin 2x$.

For the third remainder term, I need the fourth derivative:

$$f'(x) = 2 \cos 2x, \quad f''(x) = -4 \sin 2x, \quad f'''(x) = -8 \cos 2x, \quad f^{(4)}(x) = 16 \sin 2x.$$

The Remainder Term is

$$R_3(x; 1) = \frac{16 \sin 2z}{4!} (x - 1)^4.$$  

$z$ is a number between $x$ and 1. □

**Example.** Compute the Remainder Term $R_n(x; 3)$ for $f(x) = e^{4x}$.

Since I want the $n^{th}$ Remainder Term, I need to find an expression for the $(n + 1)^{st}$ derivative. I’ll compute derivative until I see a pattern:

$$f'(x) = 4e^{4x}, \quad f''(x) = 4^2 e^{4x}, \quad f'''(x) = 4^3 e^{4x}.$$  

Notice that it’s easier to see the pattern if you don’t multiply out the power of 4. Thus,

$$f^{(n)}(x) = 4^ne^{4x}, \quad \text{so} \quad f^{(n+1)}(x) = 4^{n+1} e^{4x}.$$  

The Remainder Term is

$$R_n(x; 3) = \frac{4^{n+1} e^{4z}}{(n + 1)!} (x - 3)^{n+1}.$$
There are several things you might do with the Remainder Term:

1. Estimate the error in using $p_n(x; c)$ to estimate $f(x)$ on a given interval $(c - r, c + r)$. (The interval and the degree $n$ are fixed; you want to find the error.)

2. Find the smallest value of $n$ for which $p_n(x; c)$ approximates $f(x)$ to within a given error (“tolerance”) on a given interval $(c - r, c + r)$. (The interval and the error are fixed; you want to find the degree.)

3. Find the largest interval $(c - r, c + r)$ on which $p_n(x; c)$ approximates $f(x)$ to within a given error (“tolerance”). (The degree and the error are fixed; you want to find the interval.)

Example. The Maclaurin series for $\ln(1 + x)$ is

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$ 

What is the largest error which might result from using the first three terms of the series to approximate $\ln(1 + x)$, if $0 \leq x \leq 1$?

The remainder term is

$$R_n(x; 0) = \frac{f^{(n+1)}(z)}{(n + 1)!} x^{n+1},$$

I have $0 < z < x$. I want to estimate the maximum size of $|R_3(x; 0)|$. I take absolute values, because I don’t care whether the error is positive or negative, only how large it is.

$f(x) = \ln(1 + x)$, and you can check by taking derivatives that $f^{(4)}(x) = \frac{-6}{(1 + x)^4}$. Thus, $f^{(4)}(z) = \frac{-6}{(1 + z)^4}$. So

$$|R_3(x; 0)| = \left| \frac{-6}{(1 + z)^4} (x - 0)^4 \right| = \frac{1}{4} \frac{1}{(1 + z)^4} |x|^4.$$

Since I want the largest possible error, I want to see how large the terms $\frac{1}{(1 + z)^4}$ and $|x|^4$ could be. Remember that $z$ is between 0 and $x$, and $0 \leq x \leq 1$. So

$$0 < z < x \leq 1.$$

First, $0 \leq x \leq 1$ means that $|x|^4 \leq 1^4 = 1$.

How large can $\frac{1}{(1 + z)^4}$ be, given that $0 < z < 1$? As $z$ goes from 0 to 1, $\frac{1}{(1 + z)^4}$ decreases, so it is largest if $z = 0$. This means that

$$\frac{1}{(1 + z)^4} \leq 1.$$ 

You can also see this by doing the algebra:

$$0 < z < 1$$
$$1 < z + 1 < 2$$
$$1 < (z + 1)^4 < 16$$
$$1 > \frac{1}{(1 + z)^4} \geq \frac{1}{16}.$$
In general, to estimate the $z$-term you’d have to find the absolute max on the interval for $z$. If you know that the $z$-term is either increasing or decreasing, you can check its value at the interval endpoints, and take the largest.

Using the estimates for $\frac{1}{(1+z)^2}$ and $|x|^4$, I have

$$|R_3(x;0)| \leq \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4}.$$  

The error is no greater than $\frac{1}{4}$.

I can check this by plotting the difference between the $3^{rd}$ degree Taylor polynomial and $\ln(1 + x)$.

![Graph](image)

From the picture, it looks as though the maximum error is around 0.15 (in absolute value). The estimated error was pretty conservative.  

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**Example.** (a) Compute $R_3(x;0)$ for $f(x) = \frac{1}{2 + x}$, and express $f(x)$ using $p_3(x)$ and the remainder term.

(b) Use $R_3(x;0)$ to approximate the largest error that occurs in using $p_3(x)$ to approximate $\frac{1}{2 + x}$ for $0 \leq x \leq 1$.

(a) Since I want $R_3(x;0)$, I need the fourth derivative:

$$f'(x) = -\frac{1}{(2+x)^2}, \quad f''(x) = \frac{2}{(2+x)^3}, \quad f'''(x) = \frac{-6}{(x+x)^4}, \quad f^{(4)}(x) = \frac{24}{(2+x)^5}.$$  

Thus,

$$R_3(x;0) = \frac{24}{(2+z)^5} \cdot \frac{1}{4!} x^4 = \frac{x^4}{(2+z)^5}.$$  

Now

$$\frac{1}{2 + x} = \frac{1}{2} \cdot \frac{1}{1 - \left(\frac{x}{2}\right)} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).$$  

Therefore,

$$\frac{1}{2 + x} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{(2+z)^5}\right).$$  

Here $z$ is between 0 and $x$.  

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(b) I have
\[ |R_3(x;0)| = \frac{1}{(2+z)^5} \cdot |x|^4. \]

I’ll estimate the \( z \) and \( x \)-terms one at a time.
Since \( 0 \leq x \leq 1 \), I have
\[ |x|^4 \leq 1^4 = 1. \]

Since \( 0 \leq x \leq 1 \) and \( z \) is between 0 and \( x \), it follows that \( 0 \leq z \leq 1 \). On this interval, \( \frac{1}{(2+z)^5} \) decreases, so it attains its largest value at \( z = 0 \). Therefore,
\[ \frac{1}{(2+z)^5} \leq \frac{1}{(2+0)^5} = \frac{1}{32}. \]

Alternatively,
\[
\begin{align*}
0 < z &< 1 \\
2 < 2 + z &< 3 \\
32 < (2 + z)^5 &< 243 \\
\frac{1}{32} &> \frac{1}{(2+z)^5} > \frac{1}{243}
\end{align*}
\]

Thus,
\[ |R_3(x;0)| \leq \frac{1}{32} \cdot 1 = \frac{1}{32}. \]

The error is no greater than \( \frac{1}{32} \).  \( \square \)

**Example.** Find the smallest value of \( n \) for which the \( n \)-th degree Taylor series for \( f(x) = e^{2x} \) at \( c = 0 \) approximates \( e^{2x} \) on the interval \( 0 \leq x \leq 0.3 \) with an error no greater than \( 10^{-6} \).

Notice that
\[ f'(x) = 2e^{2x}, \quad f''(x) = 2^2e^{2x}, \quad f^{(3)}(x) = 2^3e^{2x}, \quad \ldots, \quad f^{(n)}(x) = 2^ne^{2x}. \]

So
\[ |R_n(x;0)| = \left| \frac{2^{n+1}e^{2z}}{(n+1)!}x^{n+1} \right| = \frac{2^{n+1}e^{2z}}{(n+1)!} |x|^{n+1} \text{ for } 0 \leq z \leq x \leq 0.3. \]

First, I’ll estimate how large the \( z \) and \( x \)-terms can be. Since \( 0 \leq x \leq 0.3 \) and \( x^n \) is an increasing function, I have
\[ |x|^{n+1} \leq 0.3^{n+1}. \]

Since \( 0 \leq z \leq 0.3 \) and since \( e^{2z} \) is an increasing function, I have
\[ e^{2z} \leq e^{0.6}. \]

Thus,
\[ |R_n(x;0)| \leq \frac{2^{n+1}e^{0.6}}{(n+1)!} \cdot 0.3^{n+1} = e^{0.6} \frac{0.6^{n+1}}{(n+1)!}. \]

Therefore, I want the smallest \( n \) for which
\[ e^{0.6} \frac{0.6^{n+1}}{(n+1)!} < 10^{-6}. \]
I can’t solve this inequality algebraically, so I’ll have to use trial-and-error:

\[
\begin{array}{|c|c|}
\hline
n & e^{0.6} \frac{0.6^{n+1}}{(n+1)!} \\
\hline
1 & 0.32798 \ldots \\
2 & 0.06559 \ldots \\
3 & 0.00983 \ldots \\
4 & 0.00118 \ldots \\
5 & 1.18073 \ldots \cdot 10^{-4} \\
6 & 1.01205 \ldots \cdot 10^{-5} \\
7 & 7.59042 \ldots \cdot 10^{-7} \\
\hline
\end{array}
\]

The smallest value of \( n \) is \( n = 7 \). 

You can also use the Remainder Term to estimate the error in using a Taylor polynomial to approximate an integral.

**Example.** Calvin wants to impress Phoebe Small by using the MacLaurin series for \( e^{2x} \) to approximate \( \int_{0}^{0.5} xe^{2x} \, dx \) to within 0.0001. How many terms of the series should he use?

The Maclaurin series for \( e^{2x} \) is

\[ e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}. \]

(Substitute \( u = 2x \) in the standard series for \( e^u \).) I want to know how many terms of the series to use to approximate the integral.

Since \( f(x) = e^{2x} \), I have

\[ f'(x) = 2e^{2x}, \quad f''(x) = 2^2e^{2x}, \ldots f^n(x) = 2^n e^{2x}. \]

Therefore,

\[ R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)(x-c)^{n+1} = \frac{1}{(n+1)!} \cdot 2^n \cdot e^{2z} \cdot x^{n+1}. \]

In the integral, \( x \) goes from 0 to 0.5, and \( z \) is a number between 0 (the expansion point) and \( x \). Therefore, I know that \( z \) is a number between 0 and 0.5. Taking the worst possible case, the largest \( e^{2z} \) could be is \( e^{2 \cdot 0.5} = e \). Replace \( e^{2z} \) with \( e \) to obtain

\[ R_n(x) \leq \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+1}. \]

Insert this into the integral (remembering to multiply by \( x \)):

\[ \text{error} \leq \int_{0}^{0.5} \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+2} \, dx = \frac{1}{(n+1)!} 2^{n+1} \cdot e \cdot \frac{1}{n+3} \cdot (0.5)^{n+3}. \]

I want the smallest value of \( n \) for which this ugly mess is less than 0.0001. The easiest way to do this is by trial: Plug in successive values of \( n \).
\begin{align*}
\begin{array}{|c|c|c|}
\hline
n & \frac{2^{n+1}}{(n+1)!} \cdot \frac{e}{n+3} \cdot 0.5^{n+3} & n = 6 \text{ is the smallest value that works.} \quad \square \\
\hline
0 & 0.226523485\ldots & \\
1 & 0.084946307\ldots & \\
2 & 0.022652348\ldots & \\
3 & 0.004719239\ldots & \\
4 & 0.000800903\ldots & \\
5 & 0.000117980\ldots & \\
6 & 0.000014981\ldots & \\
\hline
\end{array}
\end{align*}