Separation of Variables

Separation of variables is a method for solving a differential equation. I’ll illustrate with some examples.

**Example.** Solve \( \frac{dy}{dx} = 2xy. \)

“Solve” usually means to find \( y \) in terms of \( x \). In general, I’ll be satisfied if I can eliminate the derivative by integration.

First, I rearrange the equation to get the \( x \)'s on one side and the \( y \)'s on the other (separation):

\[
\frac{dy}{y} = 2x \, dx.
\]

This is a *formal* manipulation, since I’m temporarily treating \( \frac{dy}{dx} \) as a quotient of \( dy \) by \( dx \). (See the remark below.)

Next, I integrate both sides:

\[
\int \frac{dy}{y} = \int 2x \, dx
\]

\[
\ln |y| = x^2 + C
\]

I only need an arbitrary constant on one side of the equation. Finally, I *solve* for \( y \) in terms of \( x \), if possible:

\[
e^{\ln |y|} = e^{x^2 + C}
\]

\[
|y| = e^C e^{x^2}
\]

Here’s a convenient trick which I’ll use in these situations. Think of \( |y| \) as \( \pm y \). Move the \( \pm \) to the other side:

\[
y = \mp e^C e^{x^2}.
\]

Now *define* \( C_0 = \mp e^C \):

\[
y = C_0 e^{x^2}.
\]

The last step makes the equation nicer, and it’s easier to solve for the arbitrary constant when you have an *initial value problem*. \( \Box \)

**Remark.** Here’s a justification for the formal manipulation with \( dx \) and \( dy \). Think of \( x \) and \( y \) as depending on a third variables \( t \), so \( x = f(t) \) and \( y = g(t) \). By the Chain Rule,

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.
\]

The initial equation becomes

\[
\frac{dy}{dx} = 2xy
\]

\[
\frac{dy}{dt} = 2xy
\]

\[
\frac{dt}{dx} = 2xy
\]

\[
y \frac{dy}{dt} = 2x \frac{dx}{dt}
\]
Then integrate both sides with respect to \( t \).

\[
\int y \frac{dy}{dt} dt = \int 2x \frac{dx}{dt} dt
\]

\[
\int y \, dy = \int 2x \, dx
\]

Then continue as above. In the example that follows, I'll just work formally with \( dx \) and \( dy \).

**Example.** Solve \( \frac{dy}{dx} = \frac{x}{y} + \frac{1}{y} \), where \( y(2) = 4 \).

Separate:

\[
\frac{dy}{dx} = \frac{1}{y}(x + 1)
\]

\[
y \, dy = (x + 1) \, dx
\]

Integrate:

\[
\int y \, dy = \int (x + 1) \, dx
\]

\[
\frac{1}{2} y^2 = \frac{1}{2} x^2 + x + C
\]

In this case, solving would produce plus and minus square roots, so I'll leave the equation as is. Plug in the initial condition: When \( x = 2 \), \( y = 4 \):

\[
\frac{1}{2} \cdot 4^2 = \frac{1}{2} \cdot 2^2 + 2 + C
\]

\[
8 = 2 + 2 + C
\]

\[
C = 4
\]

Hence, the solution is

\[
\frac{1}{2} y^2 = \frac{1}{2} x^2 + x + 4. \quad \Box
\]

I’ll use separation of variables to solve the equations for **exponential growth** and **Newton’s law of cooling**.