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Sequences

An **infinite sequence** is a list of numbers. The following examples should make the idea clear.

Example. Here is a familiar sequence:

$$1, 2, 4, 8, 16, \ldots, 2^n, \ldots$$

Sequences are often written using subscript notation. This one might be written

$$a_n = 2^n, \quad n = 0, 1, \ldots.$$

The a is just a dummy variable. The subscript n is the important thing, since it keeps track of the number of the term and also occurs in the formula 2^n.

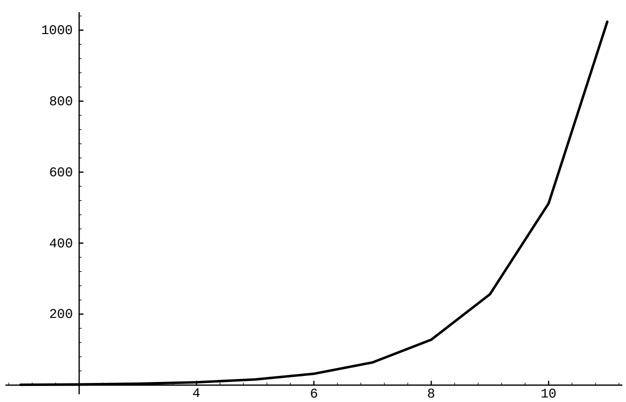
You can also write $\{2^n\}_{n=0}^{\infty}$.

There is no reason why you have to start indexing at 0. Here is the same sequence, indexed from 1:

$$b_n = 2^{n-1}, \quad n = 1, 2, \ldots.$$

The picture below shows a plot of the first few terms of the sequence using n on the horizontal axis and the value of the sequence on the vertical axis. That is, I plotted the points

$$(0, 1), (1, 2), (2, 4), (3, 8), (4, 16), \ldots.$$



To make it look like an ordinary graph, I connected the dots with segments, but you can also plot the points by themselves. □

Example. The order of the numbers in a sequence is important:

$$0, 1, 0, 0, 0, \ldots \quad \text{and} \quad 1, 0, 0, 0, 0, \ldots$$

are different sequences. □

Example. (**Arithmetic sequences**) These are sequences like

$$a_n = 7 + 3n, \quad n = 0, 1, 2, \ldots \quad \text{that is,} \quad 7, 10, 13, 16, \ldots,$$

$$b_n = 5 - 2n, \quad n = 0, 1, 2, \ldots \quad \text{that is,} \quad 5, 3, 1, -1, -3, \ldots.$$

In an arithmetic sequence, you get the next term by adding a fixed number to the preceding term. □

Example. (**Geometric sequences**) Another way to generate a sequence of numbers is to *multiply* the last number by a fixed number. For example,

$$a_n = 2^n, \quad n = 0, 1, 2, \ldots \quad \text{that is,} \quad 1, 2, 4, 8, \ldots,$$

$$b_n = \left(-\frac{1}{3}\right)^n, \quad n = 0, 1, 2, \ldots \quad \text{that is,} \quad 1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots.$$

By experimenting, you can see that the terms in a geometric sequence can do the following things:

a. The terms can go to $+\infty$. $\{2^n\}$ is an example.

b. The terms can diverge by oscillation. $\{(-3)^n\}$ diverges to $\pm\infty$ by oscillation. $\{(-1)^n\}$ diverges to ± 1 by oscillation.

c. The terms can converge to 0 or 1. $\left\{\left(-\frac{1}{3}\right)^n\right\}$ converges to 0. 1^n converges to 1.

Here are the rules. If your sequence is $a_n = r^n$, then:

a. If $r > 1$, then $a_n \to +\infty$.

b. If $r < -1$, then a_n diverges to $\pm\infty$ by oscillation.

c. If $r = 1$, then $a_n = 1$ for all n, and $a_n \to 1$.

d. If $r = -1$, then a_n diverges to ± 1 by oscillation.

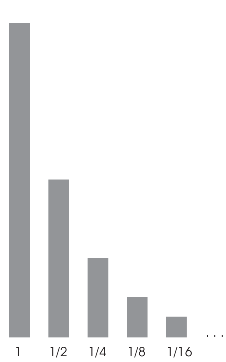
e. If $|r| < 1$, then $a_n \to 0$.

I'm being a bit informal in our use of "converges" and "diverges". I'll explain in more detail in the next example. □

Example. Here is a geometric sequence in which each term is $\frac{1}{2}$ times the previous term:

$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots.$$

The terms appear to approach 0, so it's natural to write $\lim_{n \to \infty} \frac{1}{2^n} = 0$.
Here is a picture of the terms in this sequence.



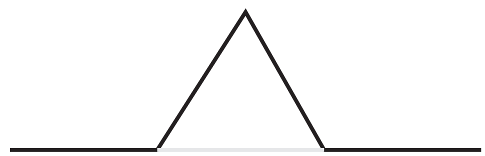
Notice that the rectangles' heights approach 0.

Here is a geometric sequence in which each term is $\frac{4}{3}$ times the previous term:

$$1, \quad \frac{4}{3}, \quad \frac{16}{9}, \quad \frac{64}{27}, \ldots, \frac{4^n}{3^n}, \ldots.$$

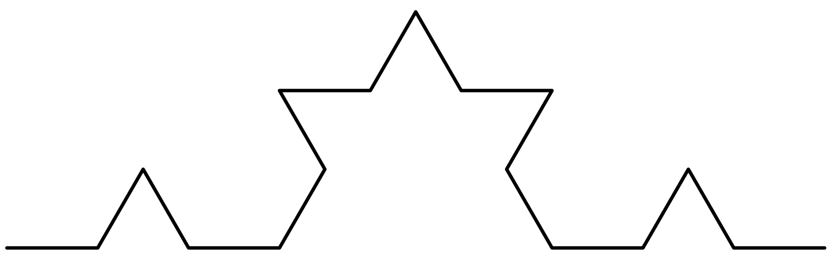
The terms appear to increase indefinitely, so I'll write $\lim_{n\to\infty} \frac{4^n}{3^n} = \infty$.

Here is an interesting way to picture of the terms in this sequence. Take a segment and divide it into thirds. Replace the middle third with a "bump" shaped like an equilateral triangle.



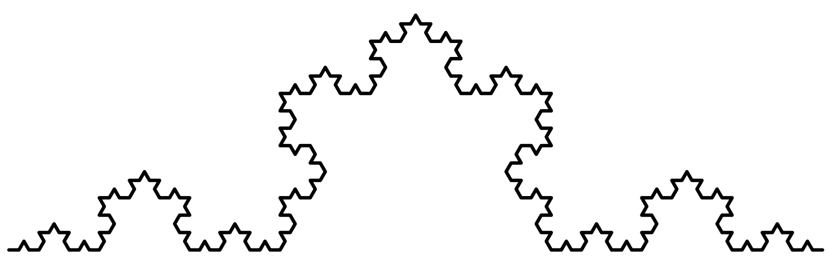
If the original segment had length 1, the new path with the triangular bump has length $\frac{4}{3}$.

Now repeat the process with each of the four segments:



Since each segment's length is multiplied by $\frac{4}{3}$, this path has total length $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

Here's the result of repeating the process two more times:



If you continue this process indefinitely, the limiting path must have infinite length, since $\lim_{n\to\infty} \frac{4^n}{3^n} = \infty$. The limiting path is an example of a **self-similar fractal**.

One of the most important questions you can ask about a sequence is: What do the terms a_n do as n gets large?

If $\{a_n\}$ is a sequence, then

$$\lim_{n\to\infty} a_n = L$$

means that you can make the terms as close to L as you please by making n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, the sequence **converges**; if $\lim_{n\to\infty} a_n$ does not exist, the sequence **diverges**.

If a sequence diverges, but the terms either increase or decrease indefinitely, then you write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty, \quad \text{respectively.}$$

Thus, $\lim_{n\to\infty} a_n = \infty$ means that you can make the terms as large (and positive) as you want by making n sufficiently large. A similar definition applies for $\lim_{n\to\infty} a_n = -\infty$.

With this definition, all the ordinary rules for computing limits apply.

Theorem. Suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences. Then:

(a) $\lim_{n\to\infty} k = k$, where k is a constant.

(b) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

(c) $\lim_{n\to\infty} a_n \cdot b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$.

(d) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$.

(e) (**Squeezing Theorem**) If $a_n \le b_n \le c_n$ for all n, and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
□

As usual, in parts (b), (c), and (d) the interpretation is that the two sides of an equation are equal when all the limits involved are defined.

Besides the rules above, you may also use L'Hôpital's Rule to compute limits of sequences.

Example. Determine whether the sequence

$$a_n = \frac{2n^2 - 3n + 1}{5 - 7n^2}, \quad n \ge 1,$$

converges or diverges. If it converges, find the limit.

$$\lim_{n\to\infty} \frac{2n^2 - 3n + 1}{5 - 7n^2} = \lim_{n\to\infty} \frac{4n - 3}{-14n} = \lim_{n\to\infty} \frac{4}{-14} = -\frac{2}{7}.$$

Hence, the series converges. □

Example. Determine whether the sequence

$$a_n = \frac{2^n + 7}{5^n - 3}, \quad n \ge 1,$$

converges or diverges. If it converges, find the limit.

Divide the top and bottom by 5^n:

$$\lim_{n\to\infty} \frac{2^n + 7}{5^n - 3} = \lim_{n\to\infty} \frac{\dfrac{2^n}{5^n} + \dfrac{7}{5^n}}{1 - \dfrac{3}{5^n}}.$$

Now $\dfrac{2^n}{5^n} = \left(\dfrac{2}{5}\right)^n$, and this goes to 0 because $\dfrac{2}{5} < 1$. Clearly $\dfrac{7}{5^n}$ and $\dfrac{3}{5^n}$ go to 0. The limit reduces to

$$\lim_{n\to\infty} \frac{\dfrac{2^n}{5^n} + \dfrac{7}{5^n}}{1 - \dfrac{3}{5^n}} = \frac{0+0}{1-0} = 0.$$

The sequence converges to 0. □

Example. (a) Determine whether the sequence

$$a_n = (-1)^n \frac{n}{n^2+1}, \quad n \ge 1,$$

converges or diverges. If it converges, find the limit.

Since $(-1)^n = \pm 1$, I have

$$-\frac{n}{n^2+1} \le (-1)^n \frac{n}{n^2+1} \le \frac{n}{n^2+1}.$$

Now

$$\lim_{n\to\infty} -\frac{n}{n^2+1} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

By the Squeezing Theorem,

$$\lim_{n\to\infty} (-1)^n \frac{n}{n^2+1} = 0. \quad \square$$

(b) Determine whether the sequence

$$b_n = (-1)^n \frac{n}{n+1}, \quad n \ge 1,$$

converges or diverges. If it converges, find the limit.

Note that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Hence, when n is large and even, $(-1)^n \dfrac{n}{n+1}$ is close to 1, and when n is large and odd, $(-1)^n \dfrac{n}{n+1}$ is close to -1. Therefore, the sequence diverges by oscillation. □

Example. Determine whether the sequence

$$a_n = \frac{(\sin n)^2}{n}$$

converges or diverges. If it converges, find the limit.

Note that

$$-1 \le \sin n \le 1, \quad \text{so} \quad 0 \le (\sin n)^2 \le 1.$$

(The "$0 \le$" comes from the fact that squares can't be negative.) Divide by n:

$$0 \le \frac{(\sin n)^2}{n} \le \frac{1}{n}.$$

Now $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, so

$$\lim_{n\to\infty} \frac{(\sin n)^2}{n} = 0,$$

by the Squeezing Theorem. The sequence converges to 0. □

Example. Here is a sequence defined by **recursion**:

$$a_0 = 1, \qquad a_{n+1} = \sqrt{12 + a_n}, \quad n \geq 1.$$

Here are the first few terms:

$$1, \sqrt{13}, \sqrt{12 + \sqrt{13}}, \sqrt{12 + \sqrt{12 + \sqrt{13}}}.$$

Assume that $\lim_{n\to\infty} a_n$ exists. What is it?
Let $L = \lim_{n\to\infty} a_n$. Then

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}.$$

So

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{12 + a_n} = \sqrt{12 + \lim_{n\to\infty} a_n}, \qquad \text{or} \qquad L = \sqrt{12 + L}.$$

(I'm assuming that it's justified to move the limit inside the square root to get the second equality, just as it would be for limits of functions.)

Now it is easy to solve for L:

$$L^2 = 12 + L, \quad L^2 - L - 12 = 0, \quad (L - 4)(L + 3) = 0.$$

Since $L \geq 0$, $L = 4$. □

Example. Start with a positive integer. If it is even, divide it by 2. If it is odd, multiply by 3 and add 1. Continue forever. You obtain a sequence of numbers — a different sequence for each number you start with.

You can use *Mathematica* to do this by **functional iteration**. Here is the function:

```
coll[n_] := (coll[n] = n/2) /; EvenQ[n] && n > 0

coll[n_] := (coll[n] = 3n + 1) /; OddQ[n] && n > 0
```

Use `NestList` to iterate the function. Here are the first 20 terms of the sequence which starts with 23:

```
NestList[coll, 23, 20]

 {23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2,
  1, 4, 2, 1, 4, 2}
```

If you try other starting numbers, you'll find that you always seem to get stuck in the $1 - 2 - 4$ loop. The **Collatz conjecture** says that this always happens. It is known to be true for starting numbers (at least) up to 3×10^{12}. □

A sequence $\{a_n\}$

- **increases** if $a_i < a_j$ whenever $i < j$.

- **decreases** if $a_i > a_j$ whenever $i < j$.

You can treat the terms of a sequence as values of a continuous function and use the first derivative to determine whether a sequence increases or decreases.

Example. Determine whether the sequence given by $a_n = \dfrac{n+1}{n+3}$ increases, decreases, or does neither.

Set $f(x) = \dfrac{x+1}{x+3}$. Then

$$f'(x) = \frac{(x+3)(1) - (x+1)(1)}{(x+3)^2} = \frac{2}{(x+3)^2}.$$

Since $f'(x) > 0$ for all x, the sequence increases. □

In some cases, it's not possible to use the derivative to determine whether a sequence increases or decreases. Here's another approach that is often useful:

- Let $\{a_n\}$ be a sequence with positive terms, and suppose that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then:

  1. If $L < 1$, the terms eventually decrease.
  2. If $L > 1$, the terms eventually increase.

Here is a rough justification for this rule. Suppose $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$. For large values of n, I have

$$\frac{a_{n+1}}{a_n} \approx L < 1, \quad \text{so} \quad a_{n+1} < a_n.$$

The last inequality says that the next term (a_{n+1}) is less than the current term (a_n), which means that the terms decrease. Similar reasoning applies if $L > 1$.

(The reason I have to say the terms *eventually* decrease or increase is that the limit tells what the sequence does for *large* values of n. For small values of n, the sequence may increase or decrease, and this behavior won't be detected by taking the limit.)

Example. Determine whether the sequence given by $a_n = \dfrac{4^n}{n!}$ increases, decreases, or does neither.

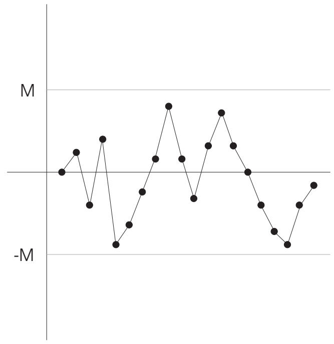
I compute $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{4^{n+1}}{(n+1)!}}{\dfrac{4^n}{n!}} = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!} =$$

$$\lim_{n\to\infty} 4 \cdot \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n \cdot (n+1)} = \lim_{n\to\infty} 4 \cdot \frac{1}{n+1} = 0.$$

Since is limit is less than 1, the terms of the sequence eventually decrease. □

A sequence $\{a_n\}$ is **bounded** if there is a number M such that $|a_n| \le M$ for all n.
Pictorially, this means that all of the terms of the sequence lie between the lines $y = -M$ and $y = M$:



I can also say a sequence is bounded if there are numbers C and D such that $C \le a_n \le D$ for all n. This definition is equivalent to the first definition. For if a sequence satisfies $|a_n| \le M$ for all n, then $-M \le a_n \le M$ for all n (so I can take $C = -M$ and $D = M$ in the second definition). On the other hand, if $C \le a_n \le D$ for all n, then $|a_n| \le \max(|C|, |D|)$, where $\max(|C|, |D|)$ is the larger of the numbers $|C|$ and $|D|$ (so I can take $M = \max(|C|, |D|)$ in the first definition).

Example. Prove that the sequence $a_n = 5 + 2\sin n$ is bounded.

Since $-1 \le \sin n \le 1$,

$$-2 \le 2\sin n \le 2, \quad \text{so} \quad 3 \le 5 + 2\sin n \le 7.$$

Thus, the sequence is bounded according to the second definition. Also, $-7 \le 5 + 2\sin n \le 7$, and hence $|5 + 2\sin n| \le 7$. Therefore, the sequence is bounded according to the first definition. □

Here's another way of telling that a sequence is bounded:

- If the terms of a sequence approach a (finite) limit, then the sequence is bounded.

To see this, suppose that $\lim_{n\to\infty} a_n = L$. By definition, this means that I can make a_n as close to L as I want by making n large enough. Suppose, for instance, I know that a_n is within 0.1 of L once n is greater than some number p. (I picked the number 0.1 at random.) Thus, all the terms after a_p are within 0.1 of L:

$$L - 0.1 < a_{p+1}, a_{p+2}, \ldots < L + 0.1.$$

What about the first p terms a_1, a_2, ..., a_p? Since there are a finite number of these terms, there must be a largest value and a smallest value among them. Suppose that the smallest value is A and the largest value is B. Thus,

$$A \le a_1, a_2, \ldots, a_p \le B.$$

Then if $C = \min(A, L - 0.1)$ is the smaller of A and $L - 0.1$ and $D = \max(B, L + 0.1)$ is the larger of B and $L + 0.1$, I must have

$$C \le a_1, a_2, \ldots, a_p, a_{p+1}, a_{p+2}, \ldots D.$$

Therefore, the sequence is bounded.

Example. Prove that the sequence $a_n = \dfrac{4n^2}{8n^2 + 3}$ is bounded.

$$\lim_{n\to\infty} \frac{4n^2}{8n^2 + 3} = \frac{1}{2}.$$

Therefore, the sequence is bounded. □