Substitution

You can use substitution to convert a complicated integral into a simpler one. In these problems, I’ll let \( u \) equal some convenient \( x \)-stuff — say \( u = f(x) \). To complete the substitution, I must also substitute for \( dx \). To do this, compute \( \frac{du}{dx} = f'(x) \), so \( du = f'(x) \, dx \). Then \( dx = \frac{du}{f'(x)} \).

Example. Compute \( \int (2x + 3)^{100} \, dx \).

\[
\int (2x + 3)^{100} \, dx = \int u^{100} \cdot \frac{du}{2} = \frac{1}{2} \int u^{100} \, du = \frac{1}{202} u^{101} + C = \frac{1}{202} (2x + 3)^{101} + C.
\]

\( u = 2x + 3, \quad du = 2 \, dx, \quad dx = \frac{du}{2} \)

\[
\int (2x + 3)^{100} \, dx = \int u^{100} \, du.
\]

Here’s what’s going on. By the Chain Rule,

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).
\]

By the definition of antiderivative,

\[
\int f'(g(x)) \cdot g'(x) \, dx = f(g(x)) + C.
\]

Now if \( u = g(x) \), I have

\[
\int f'(u) \, du = f(u) + C = f(g(x)) + C.
\]

So

\[
\int f'(g(x)) \cdot g'(x) \, dx = \int f'(u) \, du.
\]

The manipulations with \( dx \) and \( du \) are just a convenient way of doing the substitution. These are not the same “\( dx \)” and “\( du \)” we used in discussing differentials.

Example. Compute \( \int \frac{dx}{\sqrt{4 - 7x}} \).

\[
\int \frac{dx}{\sqrt{4 - 7x}} = \int \frac{1}{\sqrt{u}} \cdot \left( -\frac{du}{7} \right) = -\frac{1}{7} \int u^{-1/2} \, du = -\frac{2}{7} u^{1/2} + C = -\frac{2}{7} (4 - 7x)^{1/2} + C.
\]

\( u = 4 - 7x, \quad du = -7 \, dx, \quad dx = -\frac{du}{7} \)

Example. Later on, I’ll derive the integration formula

\[
\int \frac{dx}{x} = \ln |x| + C.
\]

Use this formula to compute \( \int \frac{1}{3x + 1} \, dx \).
\[ \int \frac{1}{3x + 1} \, dx = \int \frac{1}{u} \cdot \frac{du}{3} = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |3x + 1| + C. \]

\[ \begin{bmatrix} u = 3x + 1, \quad du = 3 \, dx, \quad dx = \frac{du}{3} \end{bmatrix} \]

**Example.** Compute \( \int x(x^2 + 5)^{50} \, dx \).

\[ \int x(x^2 + 5)^{50} \, dx = \int x u^{50} \cdot \frac{du}{2x} = \frac{1}{2} \int u^{50} \, du = \frac{1}{102} u^{51} + C = \frac{1}{102} (x^2 + 5)^{51} + C. \]

\[ \begin{bmatrix} u = x^2 + 5, \quad du = 2x \, dx, \quad dx = \frac{du}{2x} \end{bmatrix} \]

Notice that in the second step in the last example, the \( x \)'s cancelled out, leaving only \( u \)'s. If the \( x \)'s had failed to cancel, I wouldn’t have been able to complete the substitution. But what made the \( x \)'s cancel? It was the fact that I got an \( x \) from the derivative of \( u = x^2 + 5 \). This leads to the following rule of thumb.

Substitute for something whose derivative is also there.

**Example.** Compute \( \int (x + 1)^{\sqrt{x^2 + 2x + 5}} \, dx \).

\[ \int (x + 1)^{\sqrt{x^2 + 2x + 5}} \, dx = \int (x + 1)^{\sqrt{2u}} \cdot \frac{du}{2(x + 1)} = \frac{1}{2} \int \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{2/3} + C = \frac{1}{3} (x^2 + 2x + 5)^{3/2} + C. \]

\[ \begin{bmatrix} u = x^2 + 2x + 5, \quad du = (2x + 2) \, dx = 2(x + 1) \, dx, \quad dx = \frac{du}{2(x + 1)} \end{bmatrix} \]

**Example.** Compute \( \int \sin(3x + 1) \, dx \).

\[ \int \sin(3x + 1) \, dx = \int \sin u \cdot \frac{du}{3} = \frac{1}{3} \int \sin u \, du = \frac{1}{3} \cos u + C = \frac{1}{3} \cos(3x + 1) + C. \]

\[ \begin{bmatrix} u = 3x + 1, \quad du = 3 \, dx, \quad dx = \frac{du}{3} \end{bmatrix} \]

**Example.** Compute \( \int (\sin 5x)^7 \cos 5x \, dx \).

\[ \int (\sin 5x)^7 \cos 5x \, dx = \int u^7 \cos 5x \cdot \frac{du}{5 \cos 5x} = \frac{1}{5} \int u^7 \, du = \frac{1}{40} u^8 + C = \frac{1}{40} (\sin 5x)^8 + C. \]
Example. Compute $\int \frac{1}{\sqrt{x(\sqrt{x}+9)}} \, dx$.

\[
\int \frac{1}{\sqrt{x(\sqrt{x}+9)^2}} \, dx = \int \frac{1}{\sqrt{ux^2}} \cdot 2\sqrt{x} \, du = 2 \int u^{-2} \, du = -\frac{2}{u} + C = -\frac{2}{\sqrt{x}+9} + C.
\]

\[
\left[ u = \sqrt{x} + 9, \quad du = \frac{dx}{2\sqrt{x}}, \quad dx = 2\sqrt{x} \, du \right] \quad \Box
\]

Example. Compute $\int \frac{f'(x)}{(f(x)+3)^2} \, dx$.

\[
\int \frac{f'(x)}{(f(x)+3)^2} \, dx = \int \frac{f'(x)}{u^2} \cdot \frac{du}{f'(x)} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{f(x)+3} + C.
\]

\[
\left[ u = f(x)+3, \quad du = f'(x) \, dx, \quad dx = \frac{du}{f'(x)} \right] \quad \Box
\]

Example. Compute $\int \frac{\sin \frac{1}{x^2}}{x^2} \, dx$.

\[
\int \frac{\sin \frac{1}{x^2}}{x^2} \, dx = \int \sin u \cdot (-x^2 \, du) = -\int \sin u \, du = \cos u + C = \cos \frac{1}{x} + C.
\]

\[
\left[ u = \frac{1}{x}, \quad du = -\frac{dx}{x^2}, \quad dx = -x^2 \, du \right] \quad \Box
\]

The next problem introduces a new idea. In some cases, to replace the $x$’s with $u$’s, you may need to solve the substitution equation for $x$.

Example. Compute $\int (3x+4)(x+3)^{40} \, dx$.

There is no valid algebra which will allow me to multiply this out — unless I plan to multiply out $(x+3)^{40}$!

I’ll let $u = x + 3$, so $du = dx$. If I stopped with that, I’d have

\[
\int (3x+4)(x+3)^{40} \, dx = \int (3x+4)u^{40} \, du.
\]

I can’t continue as-is, because I have both $x$’s and $u$’s in the integral.

To get rid of the $x$’s, I solve the substitution equation $u = x + 3$ for $x$, to get $x = u - 3$. I can plug this into $3x+4$ to get everything in terms of $u$. Here’s the work:

\[
\int (3x+4)(x+3)^{40} \, dx = \int [3(u-3)+4]u^{40} \, du = \int (3u-5)u^{40} \, du = \int (3u^{41} - 5u^{40}) \, du = \int [u = x + 3, \quad du = dx, \quad x = u - 3]}
\]

\[
\int (3x+4)(x+3)^{40} \, dx = \int [3(u-3)+4]u^{40} \, du = \int (3u-5)u^{40} \, du = \int (3u^{41} - 5u^{40}) \, du = \int [u = x + 3, \quad du = dx, \quad x = u - 3]}
\]

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\[
\frac{3}{42} u^{42} - \frac{5}{41} u^{41} + C = \frac{3}{42}(x + 3)^{42} - \frac{5}{41}(x + 3)^{41} + C. \quad \Box
\]

**Example.** Compute \( \int \frac{4x + 7}{\sqrt{x - 2}} \, dx \).

In this problem, after making the substitution \( u = x - 2 \), I solve the substitution equation for \( x \) to get \( x = u + 2 \). Then I plug this into \( 4x + 7 \) to get rid of the \( x \)'s. Here's the work:

\[
\int \frac{4x + 7}{\sqrt{x - 2}} \, dx = \int \frac{4(u + 2) + 7}{\sqrt{u}} \, du = \int \frac{4u + 15}{\sqrt{u}} \, du = \int \left( 4\sqrt{u} + \frac{15}{\sqrt{u}} \right) \, du =
\]

\[
[u = x - 2, \quad du = dx, \quad x = u + 2]
\]

\[
\frac{8}{3} u^{3/2} + 30u^{1/2} + C = \frac{8}{3}(x - 2)^{3/2} + 30(x - 2)^{1/2} + C. \quad \Box
\]