The Tangent Plane to a Surface

The derivative of a function of one variable gives the slope of the tangent line to the graph. The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of a function of two variables $z = f(x, y)$ determine the tangent plane to the graph.

The graph of $z = f(x, y)$ is a surface in 3 dimensions. Suppose we’re trying to find the equation of the tangent plane at $(a, b, f(a, b))$.

To write down the equation of a plane, we need a point on the plane and a vector perpendicular to the plane. We have a point on the plane, namely $(a, b, f(a, b))$.

To find a vector perpendicular to the plane, we find two vectors in the plane and take their cross product. To do this, look at a small piece of the surface near the point of tangency. A small piece will be nearly flat, and will look like the parallelogram depicted below:

The vectors $\vec{a}$ and $\vec{b}$ which are the sides of the parallelogram are tangent to the surface at the point of tangency. Consider $\vec{a}$. It runs in the $x$-direction. A small change $dx$ in $x$ produces a change in $z$ — the amount the vector $\vec{a}$ “rises”.

How much does $z$ change due to a change $dx$ in $x$? The rate of change of $z$ with respect to $x$ is $\frac{\partial z}{\partial x}$, so the change in $z$ produced by changing $x$ by $dx$ is just $\frac{\partial z}{\partial x} dx$.

Now $\vec{a}$ is a vector with $x$-component $dx$, no $y$-component, and $z$-component $\frac{\partial z}{\partial x} dx$. Therefore,

$$\vec{a} = \left( dx, 0, \frac{\partial z}{\partial x} dx \right).$$

A similar argument shows that

$$\vec{b} = \left( 0, dy, \frac{\partial z}{\partial y} dy \right).$$
The cross product is
\[ \vec{a} \times \vec{b} = \left( \frac{\partial z}{\partial x} \, dx \, dy, - \frac{\partial z}{\partial y} \, dx \, dy, dx \, dy \right) = \left( - \frac{\partial z}{\partial x}, - \frac{\partial z}{\partial y}, 1 \right) \, dx \, dy. \]

I need \textit{any} vector perpendicular to the surface. Since vectors which are multiples are parallel, I may use this vector as the perpendicular vector to the surface:
\[ \left( - \frac{\partial z}{\partial x}, - \frac{\partial z}{\partial y}, 1 \right). \]

This is often referred to as the \textbf{normal vector} to the surface and denoted by \( \vec{N} \).

The tangent plane at \((a, b, f(a, b))\) is
\[ 0 \cdot (x - a) + 2 \cdot (y - b) + 1 \cdot (z - f(a, b)) = 0, \]
or \( 2y + z = 3 \).

The normal line to the surface at \((a, b, f(a, b))\) is the line which passes through \((a, b, f(a, b))\) and is perpendicular to the tangent plane. The normal line is parallel to the normal vector \( \left( - \frac{\partial z}{\partial x}, - \frac{\partial z}{\partial y}, 1 \right) \).

Therefore, the parametric equations of the normal line are
\[ x - a = - \frac{\partial f}{\partial x} \bigg|_{(a,b)} \cdot t, \quad y - b = - \frac{\partial f}{\partial y} \bigg|_{(a,b)} \cdot t, \quad z - f(a, b) = t. \]

Example. Find the equation of the tangent plane and the parametric equations of the normal line to \( z = \frac{2x}{y} - x^2 \) at \((1, 1, 1)\).

The normal vector to the surface is
\[ \left( - \frac{\partial z}{\partial x}, - \frac{\partial z}{\partial y}, 1 \right) = \left( - \frac{2}{y} + 2x, \frac{2x}{y^2}, 1 \right). \]

Plugging in \( x = 1 \) and \( y = 1 \) gives \((0, 2, 1)\).

The tangent plane is
\[ 0 \cdot (x - 1) + 2 \cdot (y - 1) + 1 \cdot (z - 1) = 0, \quad \text{or} \quad 2y + z = 3. \]

The normal line is
\[ x - 1 = 0, \quad y - 1 = 2t, \quad z - 1 = t. \]

Example. Use a tangent plane to approximate \((1.99)^2 - \frac{1.99}{1.01}\).

The idea is to think of this as the result of plugging numbers into a function \( z = f(x, y) \). What is \( f \)? Well, the form of the expression suggests that 1.99 corresponds to one of the variables and 1.01 to the other. It’s natural to use the function
\[ z = f(x, y) = x^2 - \frac{x}{y}. \]

I want to approximate \( f(1.99, 1.01) \). The point \((1.99, 1.01)\) is close to \((2, 1)\), so I’ll use the tangent plane at \((2, 1)\) to approximate \( f \).
The normal vector is 
\[ \left( -2x + \frac{1}{y}, -x, \frac{1}{y^2}, 1 \right). \]

Plug in \( x = 2, y = 1 \). This gives \((-3, -2, 1, 1)\).
When \( x = 2 \) and \( y = 1, z = 2 \). The point of tangency is \((2, 1, 2)\).

The tangent plane is 
\[ -3(x - 2) - 2(y - 1) + (z - 2) = 0, \quad \text{or} \quad z = 3x + 2y - 6. \]

Now set \( x = 1.99, y = 1.01 \). This gives \( z \approx 1.99 \). (The actual value is 1.989803.)

Here is an equivalent way to think of things that is similar to the “approximation by differentials”
technique you may have seen in first-year calculus. The change \( \Delta f \) in \( f \) produced by small changes in \( dx \) in \( x \) and \( dy \) in \( y \) is approximated by
\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \]

Thus,
\[ f(x + dx, y + dy) \approx f(x, y) + df. \]

Here \((x, y)\) denotes the “nice” point \((2, 1)\) in the last example and \((x + dx, y + dy)\) denotes the “ugly”
point \((1.99, 1.01)\) in the last example).

If you redo the example using this differential approach, you’d have
\[ f(1.99, 1.01) \approx f(2, 1) + \left( \frac{\partial f}{\partial x} \right)(dx) + \left( \frac{\partial f}{\partial y} \right)(dy) = 2 + (3)(-0.01) + (2)(0.01) = 1.99. \]

Suppose a surface is given parametrically:
\[ x = f(u, v), \quad y = g(u, v), \quad z = h(u, v). \]

Consider the vectors
\[ \vec{T}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \text{and} \quad \vec{T}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right). \]

These vectors are tangent to the curves in the surface determined by letting one of \( u \) or \( v \) vary
and holding the other constant. For example, if \( u \) varies and \( v = c \) is constant, I get the curve
\[ x = f(u, c), \quad y = g(u, c), \quad z = h(u, c). \]

The velocity vector for this curve is \( \vec{T}_u \).
Likewise, consider the curve obtained by setting \( u \) to a constant:
\[ x = f(c, v), \quad y = g(c, v), \quad z = h(c, v). \]

The velocity vector for this curve is \( \vec{T}_v \).
The cross product of $\vec{T}_u$ and $\vec{T}_v$ is a normal vector to the surface:

$$N = \vec{T}_u \times \vec{T}_v.$$ 

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**Example.** Find the equation of the tangent plane and the parametric equations for the normal line to

$$x = u^2 - v^2, \quad y = uv, \quad z = u^2 + v^2 \quad \text{at} \quad (u, v) = (2, 1).$$

First, the point of tangency is obtained by plugging $u = 2$ and $v = 1$ into $x$, $y$, and $z$. I get $x = 3$, $y = 2$, and $z = 5$. The point is $(3, 2, 5)$.

Next,

$$\vec{T}_u = (2u, v, 2u) \quad \text{and} \quad \vec{T}_v = (-2v, u, 2v).$$

When $u = 2$ and $v = 1$,

$$\vec{T}_u = (4, 1, 4) \quad \text{and} \quad \vec{T}_v = (-2, 2, 2).$$

The normal vector is

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 4 \\ -2 & 2 & 2 \end{vmatrix} = (-6, -16, 10).$$

The tangent plane is

$$-6(x - 3) - 16(y - 2) + 10(z - 5) = 0.$$ 

The normal line is

$$x - 3 = -6t, \quad y - 2 = -16t, \quad z - 5 = 10t.$$ 

\[\square\]