The **Taylor series** for \( f(x) \) at \( x = c \) is

\[
f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.
\]

(By convention, \( f^{(0)} = f \).) When \( c = 0 \), the series is called a **Maclaurin series**.

You can construct the series on the right provided that \( f \) is infinitely differentiable on an interval containing \( c \). You already know how to determine the interval of convergence of the series. However, the fact that the series converges at \( x \) does not imply that the series converges to \( f(x) \).

As an example, consider the function

\[
f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]

It is infinitely differentiable everywhere. In particular, all the derivatives of \( f \) at 0 vanish, and the Maclaurin series for \( f \) is identically 0.

Hence, the Maclaurin series for \( f \) converges for all \( x \), but only converges to \( f(x) \) at \( x = 0 \).

The following result ([1], page 418) gives a sufficient condition for the Taylor series of a function to converge to the function:

**Theorem.** Let \( f(x) \) be infinitely differentiable on \( a \leq x \leq b \), and let \( a \leq c \leq b \). Suppose there is a constant \( M \) such that \( |f^{(n)}(x)| \leq M \) for all \( n \geq 1 \), and for all \( x \) in \( N \cap [a, b] \), where \( N \) is a neighborhood of \( c \). Then for all \( x \in N \cap [a, b] \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.
\]

In other words, under reasonable conditions:

1. You can construct a Taylor series by computing the derivatives of \( f \).
2. The series will converge to \( f \) on an interval around the expansion point. (You can find the interval of convergence as usual.)

It’s tedious to have to compute lots of derivatives, and in many cases you can derive a series from another known series. Here are the series expansions for several important functions:

\[
\begin{align*}
\frac{1}{1 - u} &= \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + \cdots + u^n + \cdots & -1 < u < 1 \\
e^u &= \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots & -\infty < u < +\infty \\
\cos u &= \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots & -\infty < u < +\infty \\
\sin u &= \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots & -\infty < u < +\infty \\
\ln(1 + u) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^n}{n} + \cdots & -1 < u \leq 1 \\
(1 + u)^a &= 1 + \sum_{n=1}^{\infty} \frac{a(a - 1)\cdots(a - n + 1)}{n!} u^n & -1 < u < 1
\end{align*}
\]
**Example.** Find the Taylor series for \( \frac{1}{x+3} \) at \( a = 2 \). What is its interval of convergence?

I want things to come out in powers of \( x - 2 \), so I’ll write the function in terms of \( x - 2 \):

\[
\frac{1}{x + 3} = \frac{1}{5 + (x - 2)} \quad \text{(Make the \( x - 2 \) first)}
\]

\[
= \frac{1}{5} \cdot \frac{1}{1 + (x - 2)/5} \quad \text{(I need 5, because 5 - 2 = 3)}
\]

I’ll use the series for \( \frac{1}{1-u} \). To do this, I need \( 1 - u \) on the bottom. I make a “1” by factoring 5 out of the terms on the bottom, then I make a “−” by writing the “+” as “−”:

\[
\frac{1}{5 + (x - 2)} = \frac{1}{5} \cdot \frac{1}{1 + \frac{x - 2}{5}} = \frac{1}{5} \cdot \frac{1}{1 - \left(-\frac{x - 2}{5}\right)}
\]

Let \( u = -\frac{x - 2}{5} \) in the series for \( \frac{1}{1 - u} \). Then

\[
\frac{1}{1 - \left(-\frac{x - 2}{5}\right)} = 1 - \frac{x - 2}{5} + \left(\frac{x - 2}{5}\right)^2 - \left(\frac{x - 2}{5}\right)^3 + \cdots.
\]

Hence,

\[
\frac{1}{x + 3} = \frac{1}{5} \cdot \left[ 1 - \frac{x - 2}{5} + \left(\frac{x - 2}{5}\right)^2 - \left(\frac{x - 2}{5}\right)^3 + \cdots \right].
\]

The \( u \)-series converges for \(-1 < u < 1\), so the \( x \)-series converges for \(-1 < -\frac{x - 2}{5} < 1\), or \(-3 < x < 7\).

\[\square\]

**Example.** Find the Taylor series for \( \frac{1}{7 - x} \) at \( a = -3 \). What is its interval of convergence?

Since I’m expanding at \( a = -3 \), I need powers of \( x + 3 \):

\[
\frac{1}{7 - x} = \frac{1}{10 - (x + 3)}
\]

\[
= \frac{1}{10} \cdot \frac{1}{1 - \frac{x + 3}{10}}
\]

I let \( u = \frac{1}{10}(x + 3) \) in the series for \( \frac{1}{1 - u} \):

\[
\frac{1}{1 - \frac{1}{10}(x + 3)} = \frac{1}{10} \left( 1 + \frac{1}{10}(x + 3) + \frac{1}{10^2}(x + 3)^2 + \frac{1}{10^3}(x + 3)^3 + \cdots \right).
\]

In summation form, this is \( \frac{1}{10} \sum_{n=0}^{\infty} \frac{1}{10^n}(x + 3)^n \).
Find the interval of convergence:

\[-1 < u < 1\]

\[-1 < \frac{1}{10}(x + 3) < 1\]
\[-10 < x + 3 < 10\]
\[-13 < x < 7\]

**Example.** Find the Taylor series at \( c = 1 \) for \( e^{5x} \).

I need powers of \( x - 1 \).

\[ e^{5x} = e^{5(x-1)+5} = e^{5(x-1)} \cdot e^{5} = e^{5} \left( 1 + 5(x-1) + \frac{5^2(x-1)^2}{2!} + \frac{5^3(x-1)^3}{3!} + \cdots \right) . \]

To get this, I let \( u = 5(x-1) \) in the series for \( e^u \).

For the interval of convergence:

\[-\infty < u < \infty\]
\[-\infty < 5(x-1) < \infty\]
\[-\infty < x - 1 < \infty\]
\[-\infty < x < \infty\]

**Example.** Find the Taylor series for \( \sin x \) at \( c = \frac{\pi}{2} \).

I need powers of \( x - \frac{\pi}{2} \), so

\[ \sin x = \sin \left[ \left( x - \frac{\pi}{2} \right) + \frac{\pi}{2} \right] . \]

Next, I’ll use the angle addition formula for sine:

\[ \sin(a + b) = \sin a \cos b + \sin b \cos a . \]

I set \( a = x - \frac{\pi}{2} \) and \( b = \frac{\pi}{2} \). Since \( \cos \frac{\pi}{2} = 0 \) and \( \sin \frac{\pi}{2} = 1 \), I get

\[ \sin \left[ \left( x - \frac{\pi}{2} \right) + \frac{\pi}{2} \right] = \cos \left( x - \frac{\pi}{2} \right) = 1 - \frac{1}{2!} \left( x - \frac{\pi}{2} \right)^2 + \frac{1}{4!} \left( x - \frac{\pi}{2} \right)^4 - \frac{1}{6!} \left( x - \frac{\pi}{2} \right)^6 + \cdots . \]

**Example.** Find the Taylor series for \( \ln x \) at \( a = 1 \). What is its interval of convergence?

Use

\[ \ln(1 + u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^n}{n} + \cdots . \]

I’m expanding at \( a = 1 \), so I want the result to come out in powers of \( x - 1 \). This is easy — just set \( u = x - 1 \):

\[ \ln x = (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 + \cdots + (-1)^{n+1} \frac{1}{n} (x - 1)^n + \cdots . \]

The \( u \)-series converges for \(-1 < u \leq 1\), so the \( x \)-series converges for \(-1 < x - 1 \leq 1\), or \( 0 < x \leq 2 \).
Example. The quantity \((1 - \frac{v^2}{c^2})^{-1/2}\) occurs in special relativity. (\(v\) is the velocity of an object, and \(c\) is the speed of light.) Approximate \((1 - \frac{v^2}{c^2})^{-1/2}\) using the first two nonzero terms of the binomial series.

\[
(1 + u)^a = 1 + au + \frac{a(a - 1)}{2!}u^2 + \cdots,
\]

So for \(a = -\frac{1}{2}\),

\[
(1 + u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \cdots.
\]

Take \(u = -\frac{v^2}{c^2}\):

\[
\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \cdots \approx 1 + \frac{1}{2} \frac{v^2}{c^2}.
\]

The approximation is good as long as \(v\) is small compared to \(c\). \(\Box\)

Example. Find the Taylor series for \(\frac{x}{2 + x}\) at \(a = -1\).

Since I’m expanding at \(a = -1\), the answer must come out in terms of powers of \(x + 1\).

Start with the function you’re trying to expand. To get \(x + 1\)'s in the answer, write the given function in terms of \(x + 1\):

\[
\frac{x}{2 + x} = \frac{(x + 1) - 1}{1 + (x + 1)}.
\]

(Notice that the work has to be legal algebra.)

I’ll break up the fraction and do the pieces separately.

\[
\frac{(x + 1) - 1}{1 + (x + 1)} = \frac{x + 1}{1 + (x + 1)} - \frac{1}{1 + (x + 1)}.
\]

I want to “match” each piece against the standard series \(\frac{1}{1 - u}\). Here’s the first piece:

\[
\frac{x + 1}{1 + (x + 1)} = (x + 1) \frac{1}{1 - [-(x + 1)]}.
\]

Expand \(\frac{1}{1 - [-(x + 1)]}\) by setting \(u = -(x + 1)\) in \(\frac{1}{1 - u}\):

\[
(x + 1) \frac{1}{1 - [-(x + 1)]]} = (x + 1) \cdot (1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots) = (x + 1) - (x + 1)^2 + (x + 1)^3 - \cdots.
\]

Here’s the second piece:

\[
\frac{1}{1 + (x + 1)} = \frac{1}{1 - [-(x + 1)]} = 1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots.
\]

Put the two pieces together:

\[
[(x + 1) - (x + 1)^2 + (x + 1)^3 - \cdots] - [1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots] =
\]

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\[
\frac{(x+1)}{2} - (x+1)^2 + (x+1)^3 - \cdots = 1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots.
\]

That is,
\[
\frac{x}{2} = 1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots. \quad \square
\]

**Example.** What is the Maclaurin series for \( f(x) = 7x^2 - 3x + 13 \)? What is the Taylor series for \( f(x) = 7x^2 - 3x + 13 \) at \( a = -1 \)?

The Maclaurin series for a polynomial is the polynomial:
\[
f(x) = 7x^2 - 3x + 13.
\]

To obtain the Taylor expansion at \( a = -1 \), write the function in terms of \( x+1 \):
\[
7x^2 - 3x + 13 = 7(x+1)^2 - 17x + 6 = 7(x+1)^2 - 17(x+1) + 23.
\]

**Example.** Find \( f^{(100)}(0) \) for \( f(x) = \frac{1}{3-x} \).

The series for \( \frac{1}{3-x} \) at \( c = 0 \) is
\[
\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1 - \frac{x}{3}} = \frac{1}{3} \left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{3^n} + \cdots \right) = \\
\frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \cdots + \frac{x^n}{3^{n+1}} + \cdots.
\]

The 100th degree term is \( x^{100} \cdot \frac{1}{3^{101}} \). On the other hand, Taylor’s formula says that the 100th degree term is
\[
f^{(100)}(0) \cdot \frac{1}{100!} \cdot x^{100}. \quad \text{Equating the coefficients, I get}
\]
\[
\frac{1}{3^{101}} = \frac{f^{(100)}(0)}{100!} \quad \square
\]
\[
f^{(100)}(0) = \frac{100!}{3^{101}}
\]

While you can often use known series to find Taylor series, it’s sometimes necessary to find a series using Taylor’s formula. (In fact, that’s where the “known series” come from.)

**Example.** Find the first four nonzero terms and the general term of the Taylor series for \( f(x) = e^x \) at \( a = 0 \) and at \( a = 1 \) by computing the derivatives of \( f \).

\[
f(x) = e^x, \quad f'(x) = e^x, \quad \text{and in general} \quad f^{(n)}(x) = e^x.
\]

For \( a = 0 \), \( f^{(n)}(0) = e^0 = 1 \) for all \( n \). The Taylor series at \( a = 0 \) is
\[
f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.
\]

\[
\frac{1}{2} + (x+1)^2 + (x+1)^3 - \cdots = 1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots.
\]
For \( a = 1 \), \( f^{(n)}(1) = e \) for all \( n \). The Taylor series at \( a = 1 \) is

\[
f(x) = e + e(x - 1) + \frac{e}{2!}(x - 1)^2 + \frac{3}{3!}(x - 1)^3 + \cdots + \frac{1}{n!}(x - 1)^n + \cdots.
\]

If you truncate the series expanded at \( c \) after the \( n^{\text{th}} \)-degree term, what’s left is the \( n^{\text{th}} \)-degree Taylor polynomial \( p_n(x; c) \). For example, the third degree polynomial of \( e^x \) at \( a = 0 \) is

\[
p_3(x; 0) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.
\]

Note that the “\( n \)” here refers to the largest power of \( x \), not the number of terms. For example, the Taylor series for \( \frac{1}{1 - x^2} \) at \( a = 0 \) is

\[
\frac{1}{1 - x^2} = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots.
\]

The 2\(^{\text{nd}} \) degree Taylor polynomial and the 3\(^{\text{rd}} \) degree Taylor polynomial are equal:

\[
p_2(x; 0) = p_3(x; 0) = 1 + x^2.
\]

**Example.** Find the 3\(^{\text{rd}} \) degree Taylor polynomial for \( f(x) = \tan x \) at \( x = \frac{\pi}{4} \).

\[
f(x) = \tan x, \quad f'(x) = (\sec x)^2, \quad f''(x) = 2(\sec x)^2 \tan x, \quad f'''(x) = 2(\sec x)^4 + 4(\sec x)^2(\tan x)^2.
\]

Thus,

\[
f\left(\frac{\pi}{4}\right) = 1, \quad f'\left(\frac{\pi}{4}\right) = 2, \quad f''\left(\frac{\pi}{4}\right) = 4, \quad f'''\left(\frac{\pi}{4}\right) = 16.
\]

The 3\(^{\text{rd}} \) degree Taylor polynomial is

\[
p_3\left(x; \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3.
\]

**Example.** Suppose

\[
f(4) = 7, \quad f'(4) = -3, \quad f''(4) = 4, \quad f'''(4) = 12.
\]

Use the 3\(^{\text{rd}} \) degree Taylor polynomial for \( f \) at \( c = 4 \) to approximate \( f(4.2) \).

I have

\[
p_3(x; 4) = 7 - 3(x - 4) + \frac{4}{2!}(x - 4)^2 + \frac{12}{3!}(x - 4)^3 = 7 - 3(x - 4) + 2(x - 4)^2 + 2(x - 4)^3.
\]

Plug \( x = 4.2 \) in:

\[
f(4.2) \approx 7 - 3(4.2 - 4) + 2(4.2 - 4)^2 + 2(4.2 - 4)^3 = 6.496.
\]
It's also possible to construct power series by integrating or differentiating other power series. A *power series may be integrated or differentiated term-by-term in the interior of its interval of convergence.* (You will need to check convergence at the endpoints separately.)

**Example.** (a) Find the Taylor series at \( c = 0 \) for \( \frac{1}{8 + x} \).

(b) Find the Taylor series at \( c = 0 \) for \( \frac{1}{(8 + x)^2} \).

(a) 

\[
\frac{1}{8 + x} = \frac{1}{8} \frac{1}{1 + \frac{x}{8}} = \frac{1}{8} \frac{1}{1 - \left(-\frac{x}{8}\right)} = \frac{1}{8} \left(1 + \frac{x}{8} + \frac{x^2}{64} + \frac{x^3}{512} + \frac{x^4}{4096} + \cdots\right).
\]

\[\square\]

(b) Notice that 

\[
\frac{d}{dx} \frac{1}{8 + x} = -\frac{1}{(8 + x)^2}.
\]

Hence, 

\[
\frac{1}{(8 + x)^2} = -\frac{d}{dx} \frac{1}{8 + x} = -\frac{d}{dx} \frac{1}{8} \left(1 + \frac{x}{8} + \frac{x^2}{64} + \frac{x^3}{512} + \frac{x^4}{4096} + \cdots\right) = -\frac{1}{8} \left(-\frac{1}{8} + \frac{x}{32} - \frac{3x^2}{512} + \frac{x^3}{1024} + \cdots\right).
\]

\[\square\]

---

**Example.** (a) Find the Taylor series at \( c = 0 \) for \( \frac{1}{1 + x} \).

(b) Use the series in (a) to find the series for \( \ln(1 + u) \) expanded at \( c = 0 \).

(a) Put \( u = -x \) in the series for \( \frac{1}{1 - u} \) to obtain 

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots.
\]

It converges for \(-1 < x < 1\). \[\square\]

(b) Integrate the series in (a) from 0 to \( u \): 

\[
\ln(1 + u) = \int_0^u (1 - x + x^2 - x^3 + \cdots) \, dx = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots.
\]

This series will converge for \(-1 < u < 1\). The left side blows up at \( u = -1 \). On the other hand, if \( u = 1 \), 

\[
\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.
\]

The right side *does* converge (by the Alternating Series Test), so the \( \ln(1 + u) \) series converges for \(-1 < u \leq 1\). \[\square\]

---

**Example.** Find the Taylor series for \( \ln(5 - x) \) at \( a = 2 \).
First, note that
\[
\int_2^x \frac{1}{5-t} dt = -\ln(5-t)_2^x = -\ln(5-x) + \ln 3, \quad \text{so} \quad \ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t} dt.
\]
I integrated from 2 to \(x\) because I want the expansion at \(a = 2\).

Now find the series at \(a = 2\) for \(\frac{1}{5-t}\):
\[
\frac{1}{5-t} = \frac{1}{3-(t-2)} = \frac{1}{3} \cdot \frac{1}{1 - \frac{t-2}{3}} = \frac{1}{3} \sum_{n=0}^\infty \frac{(t-2)^n}{3^n}.
\]

Plug this series back into the integral and integrate term-by-term:
\[
\ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t} dt = \ln 3 - \frac{1}{3} \int_2^x \sum_{n=0}^\infty \frac{(t-2)^n}{3^n} dt = \ln 3 - \frac{1}{3} \sum_{n=0}^\infty \left[\frac{(t-2)^{n+1}}{3^n(n+1)}\right]_2^x =
\]
\[
\ln 3 - \frac{1}{3} \sum_{n=0}^\infty \frac{(x-2)^{n+1}}{3^n(n+1)} = \ln 3 - \sum_{n=0}^\infty \frac{(x-2)^{n+1}}{3^{n+1}(n+1)}. \quad \square
\]

**Example.** (a) Construct the Taylor series at \(c = 0\) for \(\frac{1}{1+t^2}\).

(b) Use the series in (a) to construct the Taylor series at \(c = 0\) for \(\tan^{-1} x\).

(c) Use the series in (b) to obtain a series for \(\pi\).

(a) I need powers of \(t\), so
\[
\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots. \quad \square
\]

(b) Note that
\[
\int_0^x \frac{1}{1+t^2} dt = [\tan^{-1} t]_0^x = \tan^{-1} x.
\]

Therefore,
\[
\tan^{-1} x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x (1-t^2 + t^4 - t^6 + \cdots) dt =
\]
\[
\left[ t - \frac{1}{3} t^3 + \frac{1}{5} t^5 - \frac{1}{7} t^7 + \cdots \right]_0^x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots. \quad \square
\]

(c) Plug \(x = 1\) into the series in (b), using the fact that \(\tan^{-1} 1 = \frac{\pi}{4}\):
\[
\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]
\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad \square
\]
\[
\pi = 4 \left( 1 - \frac{1}{3} + \frac{4}{5} - \frac{4}{7} + \cdots \right)
\]

8
Think of a Taylor series as a “replacement” for its function. For example, you can often use a Taylor series to compute a limit or an integral by replacing a function with its series.

**Example.** (a) Find the first 4 nonzero terms of the Taylor series at $c = 0$ for $\ln(1 + x^3)$.

(b) Use the series in (a) to guess the value of $\lim_{x \to 0} \frac{\ln(1 + x^3)}{x^3}$.

(a) Let $u = x^3$ in the series for $\ln(1 + u)$:

$$\ln(1 + x^3) = x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 + \frac{1}{4}x^{12} - \cdots.$$ 

(b) Plug the series from (a) into the limit:

$$\lim_{x \to 0} \frac{\ln(1 + x^3)}{x^3} = \lim_{x \to 0} \frac{1}{x^3} \left(x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 + \frac{1}{4}x^{12} - \cdots \right) = \lim_{x \to 0} \left(1 - \frac{1}{2}x^3 + \frac{1}{3}x^6 + \frac{1}{4}x^9 - \cdots \right) = 1.$$ 

**Example.** (a) Construct the Taylor series at $c = 0$ for $x^2e^{-x^2}$. (Write out at least the first 4 nonzero terms.)

(b) Use the first 3 terms of the series in (a) to approximate $\int_0^1 x^2e^{-x^2} \, dx$.

(c) Use the Alternating Series error estimate to estimate the error in (b).

(a) I set $u = -x^2$ in the series for $e^u$:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$ 

Multiply by $x^2$:

$$x^2e^{-x^2} = x^2 - x^4 + \frac{1}{2}x^6 - \frac{1}{6}x^8 + \frac{1}{24}x^{10} - \cdots.$$ 

(b) $\int_0^1 x^2e^{-x^2} \, dx \approx \int_0^1 \left(x^2 - x^4 + \frac{1}{2}x^6\right) \, dx = \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7\right]_0^1 = \frac{43}{210} = 0.20476 \ldots$ 

(c) The Alternating Series error estimate says that the error is less than the next term. So I take the next term in the series in (a) and integrate:

$$\int_0^1 \frac{1}{6}x^8 \, dx = \left[\frac{1}{54}x^9\right]_0^1 = \frac{1}{54}.$$ 

The error in the estimate in (b) is no greater than $\frac{1}{54} \approx 0.01851 \ldots$ 