Taylor Series

The Taylor series for \( f(x) \) at \( x = c \) is

\[
f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.
\]

(By convention, \( f^{(0)} = f \).) When \( c = 0 \), the series is called a Maclaurin series.

You can construct the series on the right provided that \( f \) is infinitely differentiable on an interval containing \( c \). You already know how to determine the interval of convergence of the series. However, the fact that the series converges at \( x \) does not imply that the series converges to \( f(x) \).

As an example, consider the function

\[
f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}
\]

It is infinitely differentiable everywhere. In particular, all the derivatives of \( f \) at 0 vanish, and the Maclaurin series for \( f \) is identically 0.

Hence, the Maclaurin series for \( f \) converges for all \( x \), but only converges to \( f(x) \) at \( x = 0 \).

The following result ([1], page 418) gives a sufficient condition for the Taylor series of a function to converge to the function:

Theorem. Let \( f(x) \) be infinitely differentiable on \( a \leq x \leq b \), and let \( a \leq c \leq b \). Suppose there is a constant \( M \) such that \( |f^{(n)}(x)| \leq M \) for all \( n \geq 1 \), and for all \( x \) in \( N \cap [a, b] \), where \( N \) is a neighborhood of \( c \). Then for all \( x \in N \cap [a, b] \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.
\]

In other words, under reasonable conditions:

1. You can construct a Taylor series by computing the derivatives of \( f \).

2. The series will converge to \( f \) on an interval around the expansion point. (You can find the interval of convergence as usual.)

It’s tedious to have to compute lots of derivatives, and in many cases you can derive a series from another known series. Here are the series expansions for several important functions:

\[
\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + \cdots + u^n + \cdots \quad -1 < u < 1
\]

\[
e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots \quad -\infty < u < +\infty
\]

\[
\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots \quad -\infty < u < +\infty
\]

\[
\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \quad -\infty < u < +\infty
\]

\[
\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^{n+1}}{n+1} + \cdots \quad -1 < u \leq 1
\]

\[
(1+u)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} u^n \quad -1 < u < 1
\]
Example. Find the Taylor series for \( \frac{1}{x+3} \) at \( a = 2 \). What is its interval of convergence?

I want things to come out in powers of \( x - 2 \), so I’ll write the function in terms of \( x - 2 \):

\[
\frac{1}{x+3} = \ldots + (x-2) \quad \text{(Make the } x - 2 \text{ first)}
\]

\[
= \frac{1}{5 + (x-2)} \quad \text{(I need 5, because } 5 - 2 = 3 \text{)}
\]

I’ll use the series for \( \frac{1}{1-u} \). To do this, I need \( 1 - u \) on the bottom. I make a “1” by factoring 5 out of the terms on the bottom, then I make a “−” by writing the “+” as “−(−)”:

\[
\frac{1}{5} \cdot \frac{1}{1 + \frac{x-2}{5}} = \frac{1}{5} \cdot \frac{1}{1 - \left(-\frac{x-2}{5}\right)}
\]

Let \( u = -\frac{x-2}{5} \) in the series for \( \frac{1}{1-u} \). Then

\[
\frac{1}{1 - \left(-\frac{x-2}{5}\right)} = 1 - \frac{x-2}{5} + \left(\frac{x-2}{5}\right)^2 - \left(\frac{x-2}{5}\right)^3 + \ldots.
\]

Hence,

\[
\frac{1}{x+3} = \frac{1}{5} \cdot \left[1 - \frac{x-2}{5} + \left(\frac{x-2}{5}\right)^2 - \left(\frac{x-2}{5}\right)^3 + \ldots \right].
\]

The \( u \)-series converges for \(-1 < u < 1\), so the \( x \)-series converges for \(-1 < -\frac{x-2}{5} < 1\), or \(-3 < x < 7\).

Example. Find the Taylor series for \( \frac{1}{7-x} \) at \( a = -3 \). What is its interval of convergence?

Since I’m expanding at \( a = -3 \), I need powers of \( x + 3 \):

\[
\frac{1}{7-x} = \frac{1}{10 - (x+3)}
\]

\[
= \frac{1}{10} \cdot \frac{1}{1 - \frac{1}{10}(x+3)}
\]

I let \( u = \frac{1}{10}(x+3) \) in the series for \( \frac{1}{1-u} \):

\[
\frac{1}{10} \cdot \frac{1}{1 - \frac{1}{10}(x+3)} = \frac{1}{10} \left(1 + \frac{1}{10}(x+3) + \frac{1}{10^2}(x+3)^2 + \frac{1}{10^3}(x+3)^3 + \cdots \right).
\]

In summation form, this is \( \frac{1}{10} \sum_{n=0}^{\infty} \frac{1}{10^n}(x+3)^n \).
Find the interval of convergence:

\[-1 < u < 1\]
\[-1 < \frac{1}{10}(x + 3) < 1\]
\[-10 < x + 3 < 10\]
\[-13 < x < 7\]

**Example.** Find the Taylor series at \(c = 1\) for \(e^{5x}\).

I need powers of \(x - 1\).

\[
e^{5x} = e^{5(x-1)+5} = e^{5(x-1)} \cdot e^5 = e^5 \left( 1 + 5(x-1) + \frac{5^2(x-1)^2}{2!} + \frac{5^3(x-1)^3}{3!} + \cdots \right).
\]

To get this, I let \(u = 5(x-1)\) in the series for \(e^u\).

For the interval of convergence:

\[-\infty < u < \infty\]
\[-\infty < 5(x-1) < \infty\]
\[-\infty < x - 1 < \infty\]
\[-\infty < x < \infty\]

**Example.** Find the Taylor series for \(\sin x\) at \(c = \frac{\pi}{2}\).

I need powers of \(x - \frac{\pi}{2}\), so

\[
\sin x = \sin \left( \left( x - \frac{\pi}{2} \right) + \frac{\pi}{2} \right).
\]

Next, I’ll use the angle addition formula for sine:

\[
\sin(a + b) = \sin a \cos b + \sin b \cos a.
\]

I set \(a = x - \frac{\pi}{2}\) and \(b = \frac{\pi}{2}\). Since \(\cos \frac{\pi}{2} = 0\) and \(\sin \frac{\pi}{2} = 1\), I get

\[
\sin \left( \left( x - \frac{\pi}{2} \right) + \frac{\pi}{2} \right) = \cos \left( x - \frac{\pi}{2} \right) = 1 - \frac{1}{2!} \left( x - \frac{\pi}{2} \right)^2 + \frac{1}{4!} \left( x - \frac{\pi}{2} \right)^4 - \frac{1}{6!} \left( x - \frac{\pi}{2} \right)^6 + \cdots.
\]

**Example.** Find the Taylor series for \(\ln x\) at \(a = 1\). What is its interval of convergence?

Use

\[
\ln(1 + u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^n}{n} + \cdots.
\]

I’m expanding at \(a = 1\), so I want the result to come out in powers of \(x - 1\). This is easy — just set \(u = x - 1\):

\[
\ln x = (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 + \cdots + (-1)^{n+1} \frac{1}{n} (x - 1)^n + \cdots.
\]

The \(u\)-series converges for \(-1 < u \leq 1\), so the \(x\)-series converges for \(-1 < x - 1 \leq 1\), or \(0 < x \leq 2\).
Example. The quantity \( \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \) occurs in special relativity. \( (v) \) is the velocity of an object, and \( c \) is the speed of light.) Approximate \( \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \) using the first two nonzero terms of the binomial series.

\[
(1 + u)^a = 1 + au + \frac{a(a - 1)}{2!} u^2 + \cdots ,
\]

So for \( a = \frac{1}{2} \),

\[
(1 + u)^{-1/2} = 1 - \frac{1}{2} u + \frac{3}{8} u^2 - \cdots .
\]

Take \( u = \frac{-v^2}{c^2} \):

\[
\left( 1 - \frac{v^2}{c^2} \right)^{-1/2} = 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \cdots \approx 1 + \frac{1}{2} \frac{v^2}{c^2} .
\]

The approximation is good as long as \( v \) is small compared to \( c \). \( \square \)

Example. Find the Taylor series for \( \frac{x}{2 + x} \) at \( a = -1 \).

Since I'm expanding at \( a = -1 \), the answer must come out in terms of powers of \( x + 1 \).

Start with the function you're trying to expand. To get \( x + 1 \)'s in the answer, write the given function in terms of \( x + 1 \):

\[
\frac{x}{2 + x} = \frac{(x + 1) - 1}{1 + (x + 1)} .
\]

(Notice that the work has to be legal algebra.)

I’ll break up the fraction and do the pieces separately.

\[
\frac{(x + 1) - 1}{1 + (x + 1)} = \frac{x + 1}{1 + (x + 1)} - \frac{1}{1 + (x + 1)} .
\]

I want to “match” each piece against the standard series \( \frac{1}{1 - u} \). Here’s the first piece:

\[
\frac{x + 1}{1 + (x + 1)} = (x + 1) \frac{1}{1 - [-(x + 1)]} .
\]

Expand \( \frac{1}{1 - [-(x + 1)]} \) by setting \( u = -(x + 1) \) in \( \frac{1}{1 - u} \):

\[
\frac{1}{1 - [-(x + 1)]} = (x + 1) \cdot (1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots) = (x + 1) - (x + 1)^2 + (x + 1)^3 - \cdots .
\]

Here’s the second piece:

\[
\frac{1}{1 + (x + 1)} = \frac{1}{1 - [-(x + 1)]} = 1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots .
\]

Put the two pieces together:

\[
\left[ (x + 1) - (x + 1)^2 + (x + 1)^3 - \cdots \right] - \left[ 1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots \right] =
\]

\[
\frac{x}{2 + x}.
\]
\[
\frac{x}{2+x} = -1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots.
\]

That is,\[
\frac{x}{2+x} = -1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots.
\]

\textbf{Example.} What is the Maclaurin series for \( f(x) = 7x^2 - 3x + 13 \)? What is the Taylor series for \( f(x) = 7x^2 - 3x + 13 \) at \( a = -1 \)?

The Maclaurin series for a polynomial is the polynomial: \( f(x) = 7x^2 - 3x + 13 \).

To obtain the Taylor expansion at \( a = -1 \), write the function in terms of \( x+1 \):

\[
7x^2 - 3x + 13 = 7(x+1)^2 - 17(x+1) + 6 = 7(x+1)^2 - 17(x+1) + 13.
\]

\textbf{Example.} Find \( f^{(100)}(0) \) for \( f(x) = \frac{1}{3-x} \).

The series for \( \frac{1}{3-x} \) at \( c = 0 \) is

\[
\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{1}{3} \left( 1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{3^n} + \cdots \right) =
\]

\[
\frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \cdots + \frac{x^n}{3^{n+1}} + \cdots.
\]

The 100\textsuperscript{th} degree term is \( \frac{x^{100}}{3^{101}} \). On the other hand, Taylor’s formula says that the 100\textsuperscript{th} degree term is \( \frac{f^{(100)}(0)}{100!} x^{100} \). Equating the coefficients, I get

\[
\frac{1}{3^{101}} = \frac{f^{(100)}(0)}{100!} \]

\[
f^{(100)}(0) = \frac{100!}{3^{101}}.
\]

While you can often use known series to find Taylor series, it’s sometimes necessary to find a series using Taylor’s formula. (In fact, that’s where the “known series” come from.)

\textbf{Example.} Find the first four nonzero terms and the general term of the Taylor series for \( f(x) = e^x \) at \( a = 0 \) and at \( a = 1 \) by computing the derivatives of \( f \).

\[
f(x) = e^x, \quad f'(x) = e^x, \quad \text{and in general} \quad f^{(n)}(x) = e^x.
\]

For \( a = 0 \), \( f^{(n)}(0) = e^0 = 1 \) for all \( n \). The Taylor series at \( a = 0 \) is

\[
f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.
\]
For \( a = 1 \), \( f^{(n)}(1) = e^{1} = e \) for all \( n \). The Taylor series at \( a = 1 \) is

\[
f(x) = e + e(x - 1) + \frac{e}{2!}(x - 1)^2 + \frac{3}{3!}(x - 1)^3 + \cdots + \frac{1}{n!}(x - 1)^n + \cdots.
\]

If you truncate the series expanded at \( c \) after the \( n \text{th} \)-degree term, what’s left is the \( n \text{th} \)-degree Taylor polynomial \( p_n(x; c) \). For example, the third degree polynomial of \( e^x \) at \( a = 0 \) is

\[
p_3(x; 0) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.
\]

Note that the “\( n \)” here refers to the largest power of \( x \), not the number of terms. For example, the Taylor series for \( \frac{1}{1-x^2} \) at \( a = 0 \) is

\[
\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots.
\]

The 2nd degree Taylor polynomial and the 3rd degree Taylor polynomial are equal:

\[
p_2(x; 0) = p_3(x; 0) = 1 + x^2.
\]

**Example.** Find the 3rd degree Taylor polynomial for \( f(x) = \tan x \) at \( x = \frac{\pi}{4} \).

\( f(x) = \tan x, \; f'(x) = (\sec x)^2, \; f''(x) = 2(\sec x)^2 \tan x, \; f'''(x) = 2(\sec x)^4 + 4(\sec x)^2(\tan x)^2. \)

Thus,

\[
f\left(\frac{\pi}{4}\right) = 1, \quad f'\left(\frac{\pi}{4}\right) = 2, \quad f''\left(\frac{\pi}{4}\right) = 4, \quad f'''\left(\frac{\pi}{4}\right) = 16.
\]

The 3rd degree Taylor polynomial is

\[
p_3\left(x; \frac{\pi}{4}\right) = 1 + 2 \left(x - \frac{\pi}{4}\right) + 2 \left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3} \left(x - \frac{\pi}{4}\right)^3.
\]

**Example.** Suppose

\[
f(4) = 7, \quad f'(4) = -3, \quad f''(4) = 4, \quad f'''(4) = 12.
\]

Use the 3rd degree Taylor polynomial for \( f \) at \( c = 4 \) to approximate \( f(4.2) \).

I have

\[
p_3(x; 4) = 7 - 3(x - 4) + \frac{4}{2!}(x - 4)^2 + \frac{12}{3!}(x - 4)^3 = 7 - 3(x - 4) + 2(x - 4)^2 + 2(x - 4)^3.
\]

Plug \( x = 4.2 \) in:

\[
f(4.2) \approx 7 - 3(4.2 - 4) + 2(4.2 - 4)^2 + 2(4.2 - 4)^3 = 6.496.
\]
It’s also possible to construct power series by integrating or differentiating other power series. A power series may be integrated or differentiated term-by-term in the interior of its interval of convergence. (You will need to check convergence at the endpoints separately.)

**Example.** (a) Find the Taylor series at $c = 0$ for $\frac{1}{8 + x}$.

(b) Find the Taylor series at $c = 0$ for $\frac{1}{(8 + x)^2}$.

(a)

\[
\frac{1}{8 + x} = \frac{1}{8} \frac{1}{1 + \frac{x}{8}} = \frac{1}{8} \frac{1}{1 - \left(\frac{x}{8}\right)} = \\
\frac{1}{8} \left(1 + \frac{x}{8} + \frac{x^2}{64} - \frac{x^3}{512} + \frac{x^4}{4096} - \cdots \right).
\]

(b) Notice that

\[
\frac{d}{dx} \frac{1}{8 + x} = -\frac{1}{(8 + x)^2}.
\]

Hence,

\[
\frac{1}{(8 + x)^2} = -\frac{d}{dx} \frac{1}{8 + x} = -\frac{d}{dx} \frac{1}{8} \left(1 - \frac{x}{8} + \frac{x^2}{64} - \frac{x^3}{512} + \frac{x^4}{4096} - \cdots \right) = \\
\frac{-1}{8} \left(-\frac{1}{8} + \frac{x}{32} - \frac{3x^2}{512} + \frac{x^3}{1024} - \cdots \right).
\]

**Example.** (a) Find the Taylor series at $c = 0$ for $\frac{1}{1 + x}$.

(b) Use the series in (a) to find the series for $\ln(1 + u)$ expanded at $c = 0$.

(a) Put $u = -x$ in the series for $\frac{1}{1 - u}$ to obtain

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots.
\]

It converges for $-1 < x < 1$. 

(b) Integrate the series in (a) from 0 to $u$:

\[
\ln(1 + u) = \int_0^u (1 - x + x^2 - x^3 + \cdots) \, dx = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots.
\]

This series will converge for $-1 < u < 1$. The left side blows up at $u = -1$. On the other hand, if $u = 1$,

\[
\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.
\]

The right side does converge (by the Alternating Series Test), so the $\ln(1 + u)$ series converges for $-1 < u \leq 1$.

**Example.** Find the Taylor series for $\ln(5 - x)$ at $a = 2$. 

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First, note that
\[ \int_2^x \frac{1}{5-t} \, dt = -\ln(5-t)|_2^x = -\ln(5-x) + \ln 3, \quad \text{so} \quad \ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t} \, dt. \]

I integrated from 2 to \( x \) because I want the expansion at \( a = 2 \).

Now find the series at \( a = 2 \) for \( \frac{1}{5-t} \):
\[ \frac{1}{5-t} = \frac{1}{3-(t-2)} = \frac{1}{3} \frac{1}{1-\frac{(t-2)}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(t-2)^n}{3^n}. \]

Plug this series back into the integral and integrate term-by-term:
\[ \ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t} \, dt = \ln 3 - \frac{1}{3} \int_2^x \sum_{n=0}^{\infty} \frac{(t-2)^n}{3^n} \, dt = \ln 3 - \frac{1}{3} \sum_{n=0}^{\infty} \frac{(t-2)^{n+1}}{3^n(n+1)} \]
\[ = \ln 3 - \frac{1}{3} \sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{3^n(n+1)}. \]

**Example.** (a) Construct the Taylor series at \( c = 0 \) for \( \frac{1}{1+t^2} \).

(b) Use the series in (a) to construct the Taylor series at \( c = 0 \) for \( \tan^{-1} x \).

(c) Use the series in (b) to obtain a series for \( \pi \).

(a) I need powers of \( t \), so
\[ \frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots. \]

(b) Note that
\[ \int_0^x \frac{1}{1+t^2} \, dt = [\tan^{-1} t]_0^x = \tan^{-1} x. \]

Therefore,
\[ \tan^{-1} x = \int_0^x \frac{1}{1+t^2} \, dt = \int_0^x \left( 1 - t^2 + t^4 - t^6 + \cdots \right) \, dt = \left[ t - \frac{1}{3} t^3 + \frac{1}{5} t^5 - \frac{1}{7} t^7 + \cdots \right]_0^x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots. \]

(c) Plug \( x = 1 \) into the series in (b), using the fact that \( \tan^{-1} 1 = \frac{\pi}{4} \):
\[ \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \]
\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \]
\[ \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots \]
Think of a Taylor series as a “replacement” for its function. For example, you can often use a Taylor series to compute a limit or an integral by replacing a function with its series.

**Example.** (a) Find the first 4 nonzero terms of the Taylor series at \( c = 0 \) for \( \ln(1 + x^3) \).

(b) Use the series in (a) to guess the values of \( \lim_{x \to 0} \frac{\ln(1 + x^3)}{x^3} \).

(a) Let \( u = x^3 \) in the series for \( \ln(1 + u) \):

\[
\ln(1 + x^3) = x^3 - \frac{1}{2} x^6 + \frac{1}{3} x^9 + \frac{1}{4} x^{12} - \cdots. \tag*{\Box}
\]

(b) Plug the series from (a) into the limit:

\[
\lim_{x \to 0} \frac{\ln(1 + x^3)}{x^3} = \lim_{x \to 0} \frac{1}{x^3} \left( x^3 - \frac{1}{2} x^6 + \frac{1}{3} x^9 + \frac{1}{4} x^{12} - \cdots \right) = \lim_{x \to 0} \left( 1 - \frac{1}{2} x^3 + \frac{1}{3} x^6 + \frac{1}{4} x^9 - \cdots \right) = 1. \tag*{\Box}
\]

**Example.** (a) Construct the Taylor series at \( c = 0 \) for \( x^2e^{-x^2} \). (Write out at least the first 4 nonzero terms.)

(b) Use the first 3 terms of the series in (a) to approximate \( \int_{0}^{1} x^2e^{-x^2} \, dx \).

(c) Use the Alternating Series error estimate to estimate the error in (b).

(a) I set \( u = -x^2 \) in the series for \( e^u \):

\[
e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.
\]

Multiply by \( x^2 \):

\[
x^2e^{-x^2} = x^2 - x^4 + \frac{1}{2} x^6 - \frac{1}{6} x^8 + \frac{1}{24} x^{10} - \cdots. \tag*{\Box}
\]

(b) The Alternating Series error estimate says that the error is less than the next term. So I take the next term in the series in (a) and integrate:

\[
\int_{0}^{1} x^2e^{-x^2} \, dx \approx \int_{0}^{1} \left( x^2 - x^4 + \frac{1}{2} x^6 \right) \, dx = \left[ \frac{1}{3} x^3 - \frac{1}{5} x^5 + \frac{1}{14} x^7 \right]_{0}^{1} = \frac{43}{210} = 0.20476 \ldots. \tag*{\Box}
\]

(b) I integrate the inequality:

\[
\int_{0}^{1} \frac{1}{6} x^8 \, dx = \left[ \frac{1}{54} x^9 \right]_{0}^{1} = \frac{1}{54}.
\]

The error in the estimate in (b) is no greater than \( \frac{1}{54} = 0.01851 \ldots. \tag*{\Box}
\]