Vector Functions

A function $f : \mathbb{R} \to \mathbb{R}^n$ is called a vector function in $\mathbb{R}^n$. We’ll focus on vector functions in the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$, but everything goes over to $\mathbb{R}^n$ without any difficulty.

**Example.** Find $f(0)$ and $f(2)$ for $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) = (t^2 + 6, \sin \pi t, \ln(t + 2)).$$

$$f(0) = (6, 0, \ln 2) \quad \text{and} \quad f(2) = (10, 0, \ln 4).$$

It is no accident that the function in the last example looked like a parametric curve, because that’s what it is. A vector function in $\mathbb{R}^n$ is a curve in $\mathbb{R}^n$.

Our main concern is the calculus of vector functions, and the basic idea is that we do everything component-by-component, and so many of the things you learned in single-variable calculus carry over with just minor adjustments.

**Definition.** Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a vector function, $c \in \mathbb{R}$, and $L \in \mathbb{R}^n$. Then $\lim_{t \to c} f(t) = L$ means: For every $\epsilon > 0$, there is a $\delta$, such that $\delta > |t - c| > 0$ implies $\epsilon > \|f(t) - L\|$.

(“$\|f(t) - L\|$” means the length of $f(t) - f(c)$, regarded as a vector in $\mathbb{R}^n$.)

**Proposition.** Suppose $f : \mathbb{R} \to \mathbb{R}^n$ has components

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

Then

$$\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_n(t)).$$

This means that the limit on the left exists if and only if all the component limits on the right exist, and in that case the two sides are equal. □

In other words, you take the limit of a vector function by taking the limit of each component. All of the usual rules for computing limits work, one component at a time.

**Example.** Find the value of $\lim_{t \to 1} \left(\frac{t^2 + 3t - 4}{t - 1}, \cos \frac{\pi t}{2}, 4e^{t-1}\right)$ if it exists.

The component functions of $f(t) = \left(\frac{t^2 + 3t - 4}{t - 1}, \cos \frac{\pi t}{2}, 4e^{t-1}\right)$ are $f_1(t) = \frac{t^2 + 3t - 4}{t - 1}$, $f_2(t) = \cos \frac{\pi t}{2}$, and $f_3(t) = 4e^{t-1}$. You can see that the component functions are ordinary one-variable functions of the kind you see in a first-term calculus course.

Note that

$$\lim_{t \to 1} \frac{t^2 + 3t - 4}{t - 1} = \lim_{t \to 1} \frac{(t + 4)(t - 1)}{t - 1} = \lim_{t \to 1} (t + 4) = 5.$$

So

$$\lim_{t \to 1} \left(\frac{t^2 + 3t - 4}{t - 1}, \cos \frac{\pi t}{2}, 4e^{t-1}\right) = (5, 0, 4).$$ □
**Definition.** Let \( f : \mathbb{R} \to \mathbb{R}^n \) be a vector function in \( \mathbb{R}^n \), and let \( c \in \mathbb{R} \). Then \( f \) is continuous at \( c \) if
\[
\lim_{t \to c} f(t) = f(c).
\]

**Proposition.** Suppose \( f : \mathbb{R} \to \mathbb{R}^n \) has components
\[
f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).
\]

Then \( f \) is continuous at \( c \) if and only if \( f_1, f_2, \ldots, f_n \) are continuous at \( c \), considered as functions \( \mathbb{R} \to \mathbb{R} \).

If that looks a bit technical, don’t worry. The meaning is that a vector function is continuous at a point if its component functions are.

**Example.** Define \( f : \mathbb{R} \to \mathbb{R}^2 \) by
\[
f(t) = \begin{cases} 
(t, t^2 + 3) & \text{if } t \neq 0 \\
(0, 2) & \text{if } t = 0
\end{cases}
\]

Prove or disprove: \( f \) is continuous at \( t = 0 \).

\[
\lim_{t \to 0} f(t) = \lim_{t \to 0} (t, t^2 + 3) = (0, 3), \quad \text{but} \quad f(0) = (0, 2).
\]

Since \( \lim_{t \to 0} f(t) \neq f(0) \), the function is not continuous at \( t = 0 \).

**Definition.** Let \( f : \mathbb{R} \to \mathbb{R}^n \) be a vector function in \( \mathbb{R}^n \). The **derivative** of \( f \) is the vector function \( f' : \mathbb{R} \to \mathbb{R}^n \) given by
\[
f'(t) = (f'_1(t), f'_2(t), \ldots, f'_n(t)).
\]

I’ll often write \( \frac{df(t)}{dt} \) or \( \frac{d}{dt} f \) for \( f'(t) \).

**Proposition.** Let \( f, g : \mathbb{R} \to \mathbb{R}^n \) be vector functions in \( \mathbb{R}^n \), and let \( c \in \mathbb{R} \). Then:

(a) If \( c = (c_1, c_2, \ldots, c_n) \) is a constant, then \( \frac{dc}{dt} = 0 \).

(b) \( \frac{d}{dt}[f(t) + g(t)] = \frac{df}{dt} + \frac{dg}{dt} \).

(c) \( \frac{d}{dt}(c \cdot f(t)) = c \cdot \frac{df}{dt} \).

(d) (Dot product) \( \frac{d}{dt}(f(t) \cdot g(t)) = \frac{df}{dt} \cdot g(t) + f(t) \cdot \frac{dg}{dt} \).

Note: In (d), all the products are dot products.

(e) (Cross product) Suppose \( n = 3 \). Then
\[
\frac{d}{dt}(f(t) \times g(t)) = \frac{df}{dt} \times g(t) + f(t) \times \frac{dg}{dt}.
\]

**Proof.** The proofs amount to proving the results component-wise. For example, consider (d). Suppose
\[
f(t) = (f_1(t), f_2(t), \ldots, f_n(t)) \quad \text{and} \quad g(t) = (g_1(t), g_2(t), \ldots, g_n(t)).
\]

Then
\[
f(t) \cdot g(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t).
\]
Using the Product Rule for functions of one variable, I have
\[
\frac{d}{dt} f(t) \cdot g(t) = \frac{d}{dt} [f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t)]
\]
\[
= [f'_1(t)g_1(t) + f_1(t)g'_1(t)] + [f'_2(t)g_2(t) + f_2(t)g'_2(t)] + \cdots + [f'_n(t)g_n(t) + f_n(t)g'_n(t)]
\]
\[
= [f'_1(t)g_1(t) + f'_2(t)g_2(t) + \cdots + f'_n(t)g_n(t)] + [f_1(t)g'_1(t) + f_2(t)g'_2(t) + \cdots + f_n(t)g'_n(t)]
\]
\[
= (f'_1(t), f'_2(t), \ldots, f'_n(t)) \cdot (g_1(t), g_2(t), \ldots, g_n(t)) + (f_1(t), f_2(t), \ldots, f_n(t)) \cdot (g'_1(t), g'_2(t), \ldots, g'_n(t))
\]
\[
= \frac{df}{dt} \cdot g(t) + f(t) \cdot \frac{dg}{dt}
\]

The other results are proved in similar fashion.

**Example.** Let
\[
f(t) = (t^2 + 3t + 17, \sin 4t).
\]
Compute \( f'(t) \) and \( f'(1) \).

\[
\begin{align*}
f'(t) &= (2t + 3, 4\cos 4t) \quad \text{and} \quad f'(1) = (5, 0, 4\cos 4)\end{align*}
\]

Thinking of \( f : \mathbb{R} \to \mathbb{R}^n \) as a curve, \( f'(t) \) is a **tangent vector** to the curve.

**Example.** Find parametric equations for the tangent line to
\[
f(t) = (t^3 + 5, (t + 1)^2, 7t + 1) \quad \text{at} \quad t = 1.
\]

The point of tangency is \( f(1) = (6, 4, 8) \). Now
\[
f'(t) = (3t^2, 2(t + 1), 7) \quad \text{so} \quad f'(1) = (3, 4, 7).
\]

Thus, \( (3, 4, 7) \) is a vector tangent to the curve, so it’s parallel to the tangent line to the curve. The tangent line is
\[
x - 6 = 3t, \quad y - 4 = 4t, \quad z - 8 = 7t.
\]

You can integrate vector functions component-by-component.

**Definition.** Suppose \( f : \mathbb{R} \to \mathbb{R}^n \) has components
\[
f(t) = (f_1(t), f_2(t), \ldots f_n(t)).
\]
Then
\[ \int f(t) \, dt = \left( \int f_1(t) \, dt, \int f_2(t) \, dt, \ldots, \int f_n(t) \, dt \right). \]

A similar definition holds for definite integrals.

**Proposition.** Let \( f, g : \mathbb{R} \to \mathbb{R}^n \) be vector functions in \( \mathbb{R}^n \), and let \( c \in \mathbb{R} \). Then:

(a) \[ \int [f(t) + g(t)] \, dt = \int f(t) \, dt + \int g(t) \, dt. \]

(b) \[ \int cf(t) \, dt = c \int f(t) \, dt. \]  

**Example.** Compute the integral \( \int (4 - (\sec t)^2, e^{6t}, 12t^2 - 8t + 5) \, dt \).

\[ \int (4 - (\sec t)^2, e^{6t}, 12t^2 - 8t + 5) \, dt = \left( 4t - \tan t, \frac{1}{6} e^{6t}, 4t^3 - 4t^2 + 5t \right) + (c_1, c_2, c_3). \]

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**Example.** Compute the integral \( \int_0^1 \left( 6t^2 + 5, \frac{3t + 1}{6 \cos 3t} \right) \, dt \).

\[ \int_0^1 \left( 6t^2 + 5, \frac{3t + 1}{6 \cos 3t} \right) \, dt = \left[ \left( 2t^3 + 5t, \frac{1}{3} \ln |3t + 1|, 2 \sin 3t \right) \right]_0^1 = \left( 7, \frac{1}{3} \ln 4, 2 \sin 3 \right). \]