Vectors

I’ll look at vectors from an algebraic point of view and a geometric point of view.

Algebraically, a vector is an ordered list of (usually) real numbers. Here are some 2-dimensional vectors:

\[(2, -3), (~\sqrt{2}, \frac{3}{5}), (0, 0).\]

The numbers which make up the vector are the vector’s components.

Here are some 3-dimensional vectors:

\[(1, 2, -17), (~\frac{1}{\sqrt{2}}, -17, \pi), (0, 0, 0).\]

Since we usually use \(x\), \(y\), and \(z\) as the coordinate variables in 3 dimensions, a vector’s components are sometimes referred to as its \(x\), \(y\), and \(z\)-components. For instance, the vector \((1, 2, -17)\) has \(x\)-component 1, it has \(y\)-component 2, and it has \(z\)-component \(-17\).

The set of 2-dimensional real-number vectors is denoted \(\mathbb{R}^2\), just like the set of ordered pairs of real numbers. Likewise, the set of 3-dimensional real-number vectors is denoted \(\mathbb{R}^3\).

Geometrically, a vector is represented by an arrow. Here are some 2-dimensional vectors:

A vector is commonly denoted by putting an arrow above its symbol, as in the picture above.

Here are some 3-dimensional vectors:

The relationship between the algebraic and geometric descriptions comes from the following fact: The vector from a point \(P(a, b)\) to a point \(Q(c, d)\) is given by \(\overrightarrow{PQ} = (c - a, d - b)\).

In 3 dimensions, the vector from a point \(P(a, b, c)\) to a point \(Q(d, e, f)\) is \(\overrightarrow{PQ} = (d - a, e - b, f - c)\).

Remark. You’ve probably already noticed the following harmless confusion: “\((3, 2)\)” can denote the point \((3, 2)\) in the \(x-y\)-plane, or the 2-dimensional real vector \((3, 2)\). Notice that the vector from the origin \((0,0)\)
to the point \((3,2)\) is the vector \((3,2)\).

So we can usually regard them as interchangeable. When there’s a need to make a distinction, I will call it out.

**Example.** (a) Find the vector from \(P(3,4,-7)\) to \(Q(-2,2,5)\).

(b) Find the vectors \(\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC},\) and \(\overrightarrow{CD}\) for the points \(A(1,1), B(2,3), C(-2,0),\) and \(D(-1,2)\).

Sketch the vectors \(\overrightarrow{AB}\) and \(\overrightarrow{CD}\).

(a) \[\overrightarrow{PQ} = (-2 - 3, 2 - 4, 5 - (-7)) = (-5, -2, 12).\]

(b) \[
\overrightarrow{AB} = (2 - 1, 3 - 1) = (1,2),
\overrightarrow{BA} = (1 - 2, 1 - 3) = (-1,-2),
\overrightarrow{AC} = (-2 - 1, 0 - 1) = (-3,-1),
\overrightarrow{CD} = (-1 - (-2), 2 - 0) = (1,2).
\]

Notice that \(\overrightarrow{BA} = -\overrightarrow{AB}\); this is true in general.

Here’s a sketch of the vectors \(\overrightarrow{AB}\) and \(\overrightarrow{CD}\):

\[\overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ are both } (1,2); \text{ in the picture, you can see that the arrows which represent the vectors have the same length and the same direction.}\]

*Geometrically,* two vectors (thought of as arrows) are **equal** if they have the same length and point in the same direction.
Example. In the picture below, assume the two lines are parallel. Which of the vectors $\vec{b}$, $\vec{c}$, $\vec{d}$ is equal to the vector $\vec{a}$?

\[ \vec{b} \text{ is not equal to } \vec{a}; \text{ it has the same direction, but not the same length.} \]

\[ \vec{c} \text{ is not equal to } \vec{a}; \text{ it has the same length, but the opposite direction.} \]

\[ \vec{d} \text{ is equal to } \vec{a}, \text{ since it has the same length and direction.} \]

Algebraically, two vectors are equal if their corresponding components are equal.

Example. Find $a$ and $b$ such that

\[ (a + 2b, a - b) = (-8, 7). \]

Set the corresponding components equal and solve for $a$ and $b$:

\[
\begin{align*}
  a + 2b &= -8 \\
  -a - b &= 7 \\
  \hline
  3b &= -15 \\
  b &= -5
\end{align*}
\]

Substituting this into $a + 2b = -8$, I get $a - 10 = -8$, so $a = 2$.

The solution is $a = 2, b = -5$. \(\square\)

The length of a geometric vector is the length of the arrow that represents it. The length of an algebraic vector is given by the distance formula. If $\vec{v} = (a, b, c)$, the length of $\vec{v}$ is

\[ ||\vec{v}|| = \sqrt{a^2 + b^2 + c^2}. \]

A vector with length 1 is called a unit vector.

Example. (a) Find the length of $(3, 12, -4)$.

(b) Show that $\left(\frac{-4}{5}, \frac{3}{5}\right)$ is a unit vector.

(a)

\[ ||(3, 12, -4)|| = \sqrt{3^2 + 12^2 + (-4)^2} = 13. \] \(\square\)
Algebraically, you add or subtract vectors by adding or subtracting corresponding components:

\[(a, b) + (c, d) = (a + c, b + d), \quad (a, b) − (c, d) = (a − c, b − d).\]

(Use an analogous procedure to add or subtract 3-dimensional vectors.) You can’t add or subtract vectors with different numbers of components. For example, you can’t add a 2 dimensional vector to a 3 dimensional vector.

Algebraically, you multiply a vector by a number by multiplying each component by the number:

\[k \cdot (a, b) = (ka, kb).\]

Vectors that are multiples are said to be parallel.

**Example.** Compute:

(a) \((1, 5) + (7, 19)\).
(b) \((2, −3, 8) − (16, 11, 0)\).
(c) \(6 \cdot (5, 3)\).
(d) \(2 \cdot (2, −1, 2) + 4 \cdot (1, −1, 3)\).

(a) 
\[ (1, 5) + (7, 19) = (1 + 7, 5 + 19) = (8, 24). \]

(b) 
\[ (2, −3, 8) − (16, 11, 0) = (2 − 16, −3 − 11, 8 − 0) = (−14, −14, 8). \]

(c) 
\[ 6 \cdot (5, 3) = (6 \cdot 5, 6 \cdot 3) = (30, 18). \]

(d) 
\[ 2 \cdot (2, −1, 2) + 4 \cdot (1, −1, 3) = (4, −2, 4) + (4, −4, 12) = (8, −6, 16). \]

Here are some properties of vector arithmetic. There is nothing surprising here.

**Proposition.** Let \(\vec{u}, \vec{v}, \) and \(\vec{w}\) be vectors (in the same space) and let \(k\) be a real number.

(a) (Associativity) \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

(b) (Commutativity) \(\vec{u} + \vec{v} = \vec{v} + \vec{u}\).

(c) (Zero vector) The vector \(\vec{0}\) with all-0 components satisfies \(\vec{0} + \vec{u} = \vec{u}\) and \(\vec{u} + \vec{0} = \vec{u}\).

(d) (Additive inverse) The additive inverse \(−\vec{u}\) of \(\vec{u}\) is the vector whose components are the negatives of the components of \(\vec{u}\). It satisfies \(\vec{u} + (−\vec{u}) = \vec{0}\).

(e) (Distributivity) \(k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}\).
Note: To say that the vectors are in the same space means that, for example, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all vectors in $\mathbb{R}^3$. But all of the results are true if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^{100}$ (100-dimensional Euclidean space).

**Proof.** The idea in all these cases is to write the vectors in component form and do the computation. For example, here is a proof of (c) in the case that $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$.

\[
\vec{0} + \vec{u} = (0, 0, 0) + (u_1, u_2, u_3) = (u_1, u_2, u_3) = \vec{u}.
\]

\[
\vec{u} + \vec{0} = (u_1, u_2, u_3) + (0, 0, 0) = (u_1, u_2, u_3) = \vec{u}.
\]

Here is a proof of (e). I’ll consider the special case where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$. Thus, $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.

Then

\[
k \cdot (\vec{u} + \vec{v}) = k \cdot [(u_1, u_2, u_3) + (v_1, v_2, v_3)]
\]

\[
= k \cdot (u_1 + v_1, u_2 + v_2, u_3 + v_3)
\]

\[
= (k \cdot (u_1 + v_1), k \cdot (u_2 + v_2), k \cdot (u_3 + v_3))
\]

\[
= (k \cdot u_1 + k \cdot v_1, k \cdot u_2 + k \cdot v_2, k \cdot u_3 + k \cdot v_3)
\]

\[
= (k \cdot u_1, k \cdot u_2, k \cdot u_3) + (k \cdot v_1, k \cdot v_2, k \cdot v_3)
\]

\[
= k \cdot \vec{u} + k \cdot \vec{v}
\]

The other parts are proved in similar fashion. 

There is an alternate notation for vectors that is often used in physics and engineering. $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors in the $x$, $y$, and $z$ directions:

\[
\hat{i} = (1, 0, 0), \quad \hat{j} = (0, 1, 0), \quad \hat{k} = (0, 0, 1).
\]

Note that

\[
(a, b, c) = a \cdot (1, 0, 0) + b \cdot (0, 1, 0) + c \cdot (0, 0, 1) = a\hat{i} + b\hat{j} + c\hat{k}.
\]

For example,

\[
(-3, 6, 10) = -3\hat{i} + 6\hat{j} + 10\hat{k}.
\]

In 2 dimensions, $(a, b) = a\hat{i} + b\hat{j}$. There is no $\hat{i}-\hat{j}-\hat{k}$ notation for vectors with more than 3 components.

You operate with vectors using the $i$-$j$-$k$ notation in the obvious ways. For example,

\[
2(3\hat{i} - 4\hat{k}) + (5\hat{i} + 7\hat{j} - 11\hat{k}) = 11\hat{i} + 7\hat{j} - 19\hat{k}.
\]

**Geometrically, multiplying a vector by a number** multiplies the length of the arrow by the number. In addition, if the number is negative, the arrow’s direction is reversed:

\[
\begin{aligned}
&\hat{a} \\
&2\hat{a} \\
&-3\hat{a}
\end{aligned}
\]
You **add** geometric vectors as shown below. Move one of the vectors — say \( \vec{b} \) — keeping its length and direction unchanged so that it starts at the end of the other vector. Since the copy has the same length and direction as the original \( \vec{b} \), it’s equal to \( \vec{b} \).

Next, draw the vector which starts at the starting point of \( \vec{a} \) and ends at the tip of \( \vec{b} \). This vector is the sum \( \vec{a} + \vec{b} \).

The picture below illustrates why the geometric addition rule follows from the algebraic addition rule. It is obviously a special case with two 2-dimensional vectors with positive components, but I think it makes the result plausible.

To add several vectors, move the vectors (keeping their lengths and directions unchanged) so that they are “head-to-tail”. In the second picture below, I moved \( \vec{b} \) and \( \vec{c} \).

Finally, draw a vector from the start of the first vector to the end of the last vector. That vector is the sum — in this case, \( \vec{a} + \vec{b} + \vec{c} \).

The picture below shows how to subtract one vector from another — in this case, \( \vec{a} - \vec{b} \) is the vector which goes from the tip of \( \vec{b} \) to the tip of \( \vec{a} \).
There are a couple of ways to see this. First, if you interpret this as an addition picture using the “head-to-tail” rule, it says

\[ \vec{b} + (\vec{a} - \vec{b}) = \vec{a}. \]

Alternatively, construct \(-\vec{b}\) by “flipping” \(\vec{b}\) around, then add \(-\vec{b}\) to \(\vec{a}\).

\[ \vec{a} + (-\vec{b}) = \vec{a} - \vec{b} \]

This gives \(\vec{a} + (-\vec{b}) = \vec{a} - \vec{b}\). As the picture shows, it is the same as the vector from the head of \(\vec{b}\) to the head of \(\vec{a}\), because the two vectors are opposite sides of a parallelogram.

**Example.** Vectors \(\vec{a}\) and \(\vec{b}\) are shown in the picture below.

Draw pictures of the vectors \(2\vec{a}\), \(3\vec{a} + 2\vec{b}\), and \(2\vec{a} - \vec{b}\).  

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