Volumes by Slicing

Suppose a solid extends from \( x = a \) to \( x = b \). Suppose that when it is cut by planes perpendicular to the \( x \)-axis, the cross-section of the solid cut by such a plane has area \( A(x) \). As usual, I divide the interval from \( a \) to \( b \) into subintervals of width \( \Delta x \) (say \( \Delta x = \frac{b-a}{n} \) for some \( n \)).

On a typical subinterval, I have cross-sections of areas \( A(x) \) and \( A(x + \Delta x) \). It’s reasonable to suppose that if the function \( A(x) \) is “nice enough”, there should be a number \( \bar{x} \) between \( x \) and \( x + \Delta x \) such that the volume of the small slice of thickness \( \Delta x \) from \( x \) to \( x + \Delta x \) is exactly

\[
A(\bar{x}) \cdot \Delta x.
\]

Adding up the volumes of such slices gives the volume of the solid:

\[
V = \sum A(\bar{x}) \cdot \Delta x.
\]

Replacing \( A(\bar{x}) \) with \( A(x) \) only gives an approximation:

\[
V \approx \sum A(x) \cdot \Delta x.
\]

But if I take the limit as \( \Delta x \to 0 \), then if \( A(x) \) is “nice enough” (for example, continuous as a function of \( x \)), then in the limit I will get the exact volume. It will be given by

\[
V = \lim_{\Delta x \to 0} \sum A(x) \cdot \Delta x = \int_a^b A(x) \, dx.
\]

Example. The cross-sections of a solid in planes perpendicular to the \( x \)-axis have area

\[
A(x) = 6x^2 + 5.
\]

Find the volume of the solid from \( x = 0 \) to \( x = 1 \).

Since the cross-sectional area function is given, I just integrate from 0 to 1:

\[
V = \int_0^1 (6x^2 + 5) \, dx = [2x^3 + 5x]_0^1 = 7. \quad \Box
\]
In the problems that follow, you need to determine the cross-sectional area function. In many cases, it comes from an area formula from geometry. Here are some common ones.

- A square with sides of length \( s \)
- An equilateral triangle with sides of length \( s \).
- An isosceles right triangle with legs of length \( s \).
- An isosceles right triangle with hypotenuse of length \( h \).
- A semicircle of radius \( r \).
- A semicircle of diameter \( d \).

**Example.** The base of a solid is the region in the \( x-y \) plane bounded above by \( y = x^2 \) and below by the \( x \)-axis, from \( x = 0 \) to \( x = 1 \). The cross-sections in planes perpendicular to the \( x \)-axis are squares with one side lying in the \( x-y \) plane. Find the volume of the solid.

![Square Cross-Sections](image1.png)

The first picture shows some of the square cross-sections. The second picture shows the base of the solid, with the edge of one of the square cross-sections drawn. The edge of the square is \( x^2 \), so the area of the cross-section is \( (x^2)^2 \). The volume is

\[
V = \int_0^1 (x^2)^2 \, dx = \int_0^1 x^4 \, dx = \left[ \frac{1}{5} x^5 \right]_0^1 = \frac{1}{5}.
\]

**Example.** The base of a solid is the region in the \( x-y \) plane bounded above by \( y = 9 - x^2 \) and below by the \( x \)-axis. The cross-sections in planes perpendicular to the \( x \)-axis are equilateral triangles with one side lying in the \( x-y \) plane. Find the volume of the solid.
The base is bounded by \( y = 9 - x^2 \) and the \( x \)-axis. The parabola intersects the \( x \)-axis at \( x = -3 \) and \( x = 3 \).

The picture shows a typical cross-section. It's an equilateral triangle, and its side has length \( 9 - x^2 \). Hence, the area of the cross-section is \( \frac{\sqrt{3}}{4}(9 - x^2)^2 \).

The volume is
\[
V = \int_{-3}^{3} \frac{\sqrt{3}}{4}(9 - x^2)^2 \, dx = \frac{\sqrt{3}}{4} \int_{-3}^{3} (81 - 18x^2 + x^4) \, dx = \frac{\sqrt{3}}{4} \left[ 81x - 6x^3 + \frac{1}{5}x^5 \right]_{-3}^{3} = \frac{324\sqrt{3}}{5} = 112.23689 \ldots \]

**Example.** The base of a solid is the region in the first quadrant cut off by the line \( x + y = 8 \). The cross-sections in planes perpendicular to the \( x \)-axis are semicircles with their diameters lying in the \( x-y \) plane. Find the volume of the solid.

The diameter of a typical cross-section is \( y = 8 - x \), so the radius is \( \frac{8 - x}{2} \). The volume is
\[
V = \int_0^8 \frac{\pi}{2} \left( \frac{8 - x}{2} \right)^2 \, dx = \frac{\pi}{8} \int_0^8 (8 - x)^2 \, dx = \frac{\pi}{8} \left[ \frac{1}{3}(8 - x)^3 \right]_0^8 = \frac{64\pi}{3} = 67.02064 \ldots \]

**Example.** The base of a solid is the region in the first quadrant cut off by the line \( x + y = 1 \). The cross-sections in planes perpendicular to the \( x \)-axis are isosceles right triangles with the hypotenuses lying in the \( x-y \) plane. Find the volume of the solid.
Since the hypotenuse of a typical triangle is $1 - x$, the side of such a triangle is $\frac{1 - x}{\sqrt{2}}$. The area of a triangular slice is one-half the base times the height, which is

$$\frac{1}{2} \left( \frac{1 - x}{\sqrt{2}} \right) \left( \frac{1 - x}{\sqrt{2}} \right) = \frac{1}{4} (1 - x)^2.$$

The volume is

$$V = \frac{1}{4} \int_0^1 (1 - x)^2 \, dx = \frac{1}{4} \left[ -\frac{1}{3} (1 - x)^3 \right]_0^1 = \frac{1}{12}. \quad \square$$

**Example.** A solid hemisphere of radius 4 has its base in the $x$-$y$-plane. It is cut by a plane parallel to the $x$-$y$-plane and 1 unit above it. Find the volume of the part of the hemisphere which lies between the cutting plane and the $x$-$y$-plane.

In this problem, you have to decide how to slice the solid in order to give cross-sections whose areas you can compute. Slicing the solid by planes parallel to the $x$-$y$ plane produces circular disks.

The next picture shows the solid in cross-section, with a typical slice drawn.

By Pythagoras’ theorem, the radius of a disk lying $z$ units above the $x$-$y$-plane is $\sqrt{16 - z^2}$, so its cross-sectional area is $\pi(16 - z^2)$.

The volume is

$$\int_0^1 \pi(16 - z^2) \, dz = \pi \left[ 16z - \frac{1}{3} z^3 \right]_0^1 = \frac{47\pi}{3} = 49.21828 \ldots. \quad \square$$