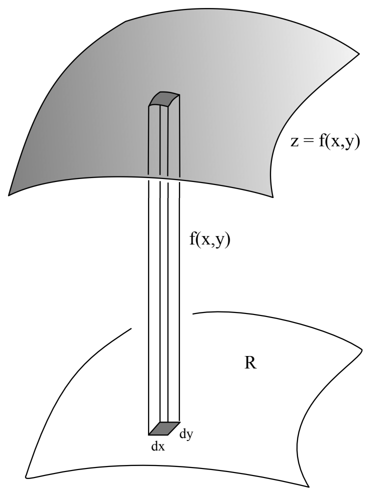
Volumes

If R is a region in the x-y plane and $z = f(x, y)$ is a function, the **volume** of the solid lying above R and below the graph of the function is given by

$$\iint_R f(x, y)\, dx\, dy.$$

The picture gives a heuristic justification for this:

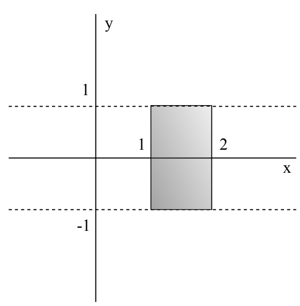


The region R is partitioned into boxes, each dx by dy. Above a box we construct a rectangular parallelepiped (i.e. a "tall box") up to the surface. The height of the box is $z = f(x, y)$, the height of the surface. The volume of a "tall box" is $f(x, y)\, dx\, dy$. The double integral "adds up" the volumes of the "tall boxes" over R to get the total volume.

A careful justification would use Riemann sums for the double integral.

This is really a **signed volume**: If the function is zero or negative on R, the integral may not represent the physical volume.

Example. Find the volume of the solid lying below the graph of $z = 24x^2y^2$ and above the following region in the x-y-plane:



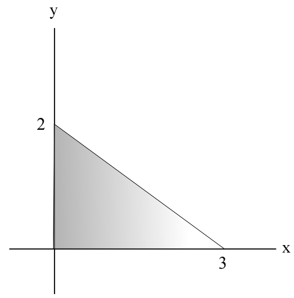
The region is

$$\left\{ \begin{array}{c} 1 \le x \le 2 \\ -1 \le y \le 1 \end{array} \right\}$$

The volume is

$$\int_{-1}^{1} \int_{1}^{2} 24x^2y^2\, dx\, dy = \int_{-1}^{1} \left[8x^3y^2\right]_1^2\, dy = \int_{-1}^{1} 56y^2\, dy = \left[\frac{56}{3}y^3\right]_{-1}^{1} = \frac{112}{3} = 37.33333\ldots. \quad \square$$

Example. Find the volume of the solid lying below the graph of $z = 18xy$ and above the following region in the x-y-plane:



First, I'll find the equation of the line, which has x-intercept 3 and y-intercept 2. Suppose the line is $ax + by = c$. The x-intercept is $(3, 0)$, so plugging this in, I get

$$3a = c, \quad \text{so} \quad a = \frac{c}{3}.$$

The y-intercept is $(0, 2)$, so plugging this in I get

$$2b = c, \quad \text{so} \quad b = \frac{c}{2}.$$

Thus,

$$\begin{aligned} \frac{c}{3}x + \frac{c}{2}y &= c \\ 2x + 3y &= 6 \\ y &= \frac{1}{3}(6 - 2x) \end{aligned}$$

The triangular region is

$$\left\{ \begin{array}{c} 0 \le x \le 3 \\ 0 \le y \le \frac{1}{3}(6 - 2x) \end{array} \right\}$$

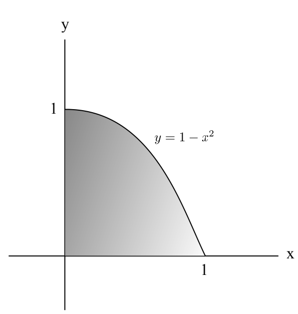
The volume is

$$\int_0^3 \int_0^{(6-2x)/3} 18xy\, dy\, dx = \int_0^3 \left[9xy^2\right]_0^{(6-2x)/3} dx = \int_0^3 x(6 - 2x)^2\, dx = \int_0^3 (36x - 24x^2 + 4x^3)\, dx =$$

$$\left[18x^2 - 8x^3 + x^4\right]_0^2 = 27. \quad \square$$

Example. Find the volume of the solid lying below the graph of $z = 3xy^2$ and above the following region in the x-y-plane:

$$R = \left\{ \begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le 1 - x^2 \end{array} \right\}$$



$$\int_0^1 \int_0^{1-x^2} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_0^{1-x^2}\,dx = \int_0^1 x(1-x^2)^3\,dx = \int_1^0 xu^3 \cdot \frac{du}{-2x} =$$

$$\left[u = 1 - x^2, \quad du = -2x\,dx, \quad dx = \frac{du}{-2x}; \quad x = 0, u = 1; x = 1, u = 0\right]$$

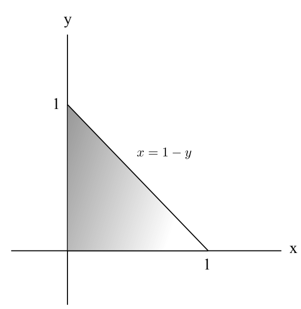
$$\frac{1}{2}\int_0^1 u^3\,du = \frac{1}{2}\left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{8}. \quad \square$$

If $f(x, y) \ge g(x, y)$ for (x, y) in a region R, the volume bounded above by the graph of f and bounded below by the graph of g, and lying inside the cylinder determined by R, is given by

$$\iint_R [f(x, y) - g(x, y)]\,dx\,dy.$$

Example. Find the volume of the solid bounded above by $z = 4y + 1$ and bounded below by $z = -1 - 2x$, and lying inside the triangular cylinder

$$\left\{ \begin{matrix} 0 \le y \le 1 \\ 0 \le x \le 1 - y \end{matrix} \right\}.$$



The volume is

$$\int_0^1 \int_0^{1-y} ((4y + 1) - (-1 - 2x))\,dx\,dy = \int_0^1 \int_0^{1-y} (2x + 4y + 2)\,dx\,dy = \int_0^1 \left[x^2 + 4xy + 2x\right]_0^{1-y}\,dy =$$

$$\int_0^1 \left((1-y)^2 + 4y(1-y) + 2(1-y)\right)\,dy = \int_0^1 (-3y^2 + 3)\,dy =$$

$$\left[-y^3 + 3y\right]_0^1 = 2. \quad \square$$
