Math 345/504
1–26–2018

Solutions to Problem Set 1

1. (a) Write the group element $x^5y^{-2}z^3$ using additive notation. (Assume the operation is commutative.)

(b) Write the group element $-2x + 11y - 4z$ using multiplicative notation. (Assume the operation is commutative.)

(a) In additive notation, this is $5x - 2y + 3z$. $\square$

(b) In multiplicative notation, this is $x^{-2}y^{11}z^{-4}$. $\square$

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2. Let G be a group (with the operation written multiplicatively), and let a and b be arbitrary elements of G.

(a) Simplify $b^3a^{-1}(ab^{-2})^3b^2a^{-1}b^2$ as much as possible.
(b) Solve for x in terms of a and b, simplifying your answer as much as possible:

$$a^{-3}xa^2b = a^{-1}bab.$$

(a)

$$b^3a^{-1}(ab^{-2})^3b^2a^{-1}b^2 = b^3a^{-1}(ab^{-2})(ab^{-2})(ab^{-2})b^2a^{-1}b^2 = ba. \quad \square$$

(b)

$$a^{-3}xa^2b = a^{-1}bab$$
$$a^3a^{-3}xa^2bb^{-1}a^{-2} = a^3a^{-1}babb^{-1}a^{-2}$$
$$x = a^2ba^{-1} \quad \square$$

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3. This is the operation table for a group G, written using multiplicative notation. The identity is 1.

$\cdot$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	a	d	c	f	e	1	g	b
b	b	g	d	a	f	c	1	e
c	c	b	e	d	g	f	a	1
d	d	e	f	g	1	a	b	c
e	e	1	g	b	a	d	c	f
f	f	c	1	e	b	g	d	a
g	g	f	a	1	c	b	e	d

(a) Compute $e \cdot f$.

(b) Compute b^2.

(c) Compute b^{-1}.

(d) Give a specific example to prove that G is not abelian.

(a) $e \cdot f = c$. □

(b) $b^2 = d$. □

(c) $b^{-1} = f$. □

(d) $e \cdot f = c$, but $f \cdot e = g$, so $e \cdot f \neq f \cdot e$. □

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4. The following set is a group under multiplication mod 20:

$$G = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

So, for example, $17 \cdot 11 = 187 = 7$.

(a) Compute $11 \cdot 9$ in G.

(b) Find 7^{-1} in G.

(c) When the operation in a group G is written using multiplicative notation, the **order** of an element $g \in G$ is the smallest positive power n such that $g^n = 1$. The identity 1 is the only element with order 1.
For example, $11^2 = 1$, so the order of 11 is 2.
Find the order of 13 in G.

[Compute powers of 13, stopping at the first power of 13 which is equal to 1.]

(a) $11 \cdot 9 = 99 = 19$. □

(b) Since $7 \cdot 3 = 1$, I have $7^{-1} = 3$. □

(c)

$$13^2 = 9, \quad 13^3 = 7, \quad 13^4 = 1.$$

The order of 13 is 4. □

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5. The following set is a group under addition mod 12:

$$\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

For example,

$$3 + 7 = 10, \quad \text{but} \quad 5 + 9 = 14 = 2.$$

(a) Compute $9 + 11$ in $\mathbb{Z}_{12}$.

(b) Find -8 in $\mathbb{Z}_{12}$.

(c) When the operation in a group G is written using additive notation, the **order** of an element $g \in G$ is the smallest positive multiple n such that $n \cdot g = 0$. The identity 0 is the only element with order 1.
(Remember that "$n \cdot g$" is shorthand for $\underbrace{g + g + \cdots + g}_{n \text{ times}}$. It is **not** multiplication in the group, since the operation is addition.)
For example,

$$3 \cdot 4 = 4 + 4 + 4 = 0 \quad \text{in} \quad \mathbb{Z}_{12}.$$

Hence, the order of 4 is 3.
Find the order of 8 in $\mathbb{Z}_{12}$.

[Compute multiples of 8 and stop at the first multiple which equals 0.]

(a) $9 + 11 = 20 = 8$ in $\mathbb{Z}_{12}$. □

(b) $8 + 4 = 0$, so $-8 = 4$ in $\mathbb{Z}_{12}$. □

(c)

$$2 \cdot 8 = 4, \quad 3 \cdot 8 = 0.$$

Hence, the order of 8 is 3. □

[Math 504]

6. Let G be a group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

Suppose G is abelian. Let $a, b \in G$. Then

$$\begin{aligned} (ab)^{-1} &= b^{-1}a^{-1} && \text{(Formula for inverse of a product)} \\ &= a^{-1}b^{-1} && \text{(G is abelian)} \end{aligned}$$

Conversely, suppose that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Let $a, b \in G$. I must show that $ab = ba$. I have

$$\begin{aligned} (ab)^{-1} &= a^{-1}b^{-1} && \text{(Given)} \\ b^{-1}a^{-1} &= a^{-1}b^{-1} && \text{(Formula for inverse of a product)} \\ (b^{-1}a^{-1})^{-1} &= (a^{-1}b^{-1})^{-1} && \\ ab &= ba && \text{(Formula for inverse of a product)} \end{aligned}$$

Therefore, G is abelian. □

Courage consists of the power of self-recovery. - Ralph Waldo Emerson