

## Solutions to Problem Set 11

1. (a) List the elements of  $U_{15}$ .
- (b) Compute  $4 \cdot 7$  in  $U_{15}$ .
- (c) Compute  $13^{-1}$  in  $U_{15}$ .
- (d) List the elements of the cyclic subgroup  $\langle 8 \rangle$  in  $U_{15}$ .

(a)

$$U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}. \quad \square$$

(b)

$$4 \cdot 7 = 13. \quad \square$$

(c) Since  $13 \cdot 7 = 1$ , it follows that  $13^{-1} = 7$ .  $\square$

(d)

$$\langle 8 \rangle = \{1, 8, 4, 2\}. \quad \square$$

2. Prove directly that  $(\mathbb{Q}, +)$  is not cyclic.

I'll show that if  $x \in \mathbb{Q}$ , then there is some rational number that is not contained in  $\langle x \rangle$ . This will prove that  $\mathbb{Q} \neq \langle x \rangle$ , and hence that  $\mathbb{Q}$  is not cyclic.

First, if  $x = 0$ , then  $\langle x \rangle = \{0\}$ . Any nonzero rational number is not contained in  $\langle x \rangle$ .

Now assume  $x \neq 0$ .

$x$  and  $-x$  and both elements of  $\langle x \rangle$ , and since  $x \neq 0$  one of them must be positive. Without loss of generality, assume  $x > 0$ .

I claim that  $\frac{1}{2}x \notin \langle x \rangle$ . Suppose to the contrary that  $\frac{1}{2}x \in \langle x \rangle$ . Then for some  $n \in \mathbb{Z}$ ,

$$nx = \frac{1}{2}x.$$

Since  $x \neq 0$ , I get  $n = \frac{1}{2}$ , contradicting the fact that  $n$  is an integer.

Therefore,  $\mathbb{Q}$  is not cyclic.  $\square$

[MATH 504]

3. This is a multiplication table for  $U_{18}$ :

	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

Find all the generators of  $U_{18}$ .

Note that  $5^2 = 7$  and  $5^3 = 17$ . Hence, 5 does not have order 1, 2, or 3. Hence, 5 must have order 6, and it's a generator.

Thus,  $U_{18}$  is cyclic of order 6. The generators in  $\mathbb{Z}_6$  are 1 and 5. So the generators in  $U_{18}$  are  $5^1 = 5$  and  $5^5 = 11$ .  $\square$

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*We have to be watchful in case we goof up, but we are uncertain as to what we are going to goof up. -*  
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