Math 345/504
3–21–2018

Solutions to Problem Set 16

1. There are 50 groups of order 72. How many nonabelian groups of order 72 are there?

Note that $72 = 2^3 \cdot 3^2$. There are 3 ways to write 2^3 as a product:

$$2^3, \quad 2 \cdot 2^2, \quad 2 \cdot 2 \cdot 2.$$

There are 2 ways to write 3^2 as a product:

$$3^2, \quad 3 \cdot 3.$$

Hence, there are $3 \cdot 2 = 6$ possible primary decompositions for abelian groups of order 72. Therefore, there are $50 - 6 = 44$ nonabelian groups of order 72. □

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2. Make a list giving the invariant factor decomposition and the corresponding primary decomposition for all abelian groups of order $400 = 2^4 \cdot 5^2$.

primary decomposition	invariant factor decomposition
$\mathbb{Z}_{16} \times \mathbb{Z}_{25}$	$\mathbb{Z}_{400}$
$\mathbb{Z}_{16} \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_5 \times \mathbb{Z}_{80}$
$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_{200}$
$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{10} \times \mathbb{Z}_{40}$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$	$\mathbb{Z}_4 \times \mathbb{Z}_{100}$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{20} \times \mathbb{Z}_{20}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{100}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_{10} \times \mathbb{Z}_{20}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{50}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10}$

□

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3. Find all abelian groups of order 81 which contain an element of order 27. For each such group, give its primary decomposition and a specific element of order 27.

The abelian groups of order 81 are

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9, \quad \mathbb{Z}_9 \times \mathbb{Z}_9, \quad \mathbb{Z}_3 \times \mathbb{Z}_{27}, \quad \mathbb{Z}_{81}.$$

Elements of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ have order less than or equal to 3, because

$$3(a, b, c, d) = (3a, 3b, 3c, 3d) = (0, 0, 0, 0).$$

Elements of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$ have order less than or equal to 9, because

$$9(a, b, c) = (9a, 9b, 9c), = (0, 0, 0).$$

Elements of $\mathbb{Z}_9 \times \mathbb{Z}_9$ have order less than or equal to 9, because

$$9(a,b) = (9a, 9b), = (0,0).$$

$\mathbb{Z}_3 \times \mathbb{Z}_{27}$ has elements of order 27 — for example, $(0,1)$.
$\mathbb{Z}_{81}$ has elements of order 27 — for example, 3. □

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4. Find the primary decomposition and the invariant factor decomposition of $\mathbb{Z}_{20} \times \mathbb{Z}_{45} \times \mathbb{Z}_{84}$.

The primary decomposition is

$$\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \times \mathbb{Z}_7 \times \mathbb{Z}_4 \times \mathbb{Z}_3.$$

The invariant factor decomposition is $\mathbb{Z}_{60} \times \mathbb{Z}_{1260}$. □

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5. Find an element of $\mathbb{Z}_{20} \times \mathbb{Z}_{24}$ having the largest possible order. Explain why no element can have larger order.

If $(a,b) \in \mathbb{Z}_{20} \times \mathbb{Z}_{24}$, then $120 \cdot (a,b) = (120a, 120b) = (0,0)$. So every element of $\mathbb{Z}_{20} \times \mathbb{Z}_{24}$ has order less than or equal to 120.

The element $(1,1)$ has order $[20,24] = 120$ in $\mathbb{Z}_{20} \times \mathbb{Z}_{24}$. Therefore, it's an element of the largest possible order. □

[MATH 504]

6. Suppose G is an abelian group of order 360 whose elements of largest order have order 90. Find its invariant factor decomposition.

The invariant factor decompositions must have the form

$$\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_n} \times \mathbb{Z}_{90}.$$

Here $d_1 \mid d_2 \mid \cdots \mid d_n \mid 90$, and the product of the d's is $\dfrac{360}{90} = 4$. The only possibility is

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{90}. \quad □$$

The same road goes both up and down. - HERACLITUS