Math 345/504
4–2–2018

Solutions to Problem Set 19

1. Let H be the subset of $GL(2,\mathbb{R})$ consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \quad \text{where} \quad x \neq 0.$$

(a) Prove that H is a subgroup of $GL(2,\mathbb{R})$.

(b) Prove that H is normal.

(a) If $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \in H$, then

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} xy & 0 \\ 0 & xy \end{bmatrix} \in H.$$

Thus, H is closed under multiplication.
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$: The identity is in H.
If $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \in H$, then

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}^{-1} = \begin{bmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{bmatrix} \in H.$$

Thus, H is closed under taking inverses.
Hence, H is a subgroup of $GL(2,\mathbb{R})$. $\square$

(b) Let $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \in H$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R})$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\left(\frac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \in H.$$

Therefore, H is normal. $\square$

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2. Use the Universal Property of the Quotient to show that thefunction $f : \mathbb{Z} \to \dfrac{\mathbb{Z}}{8\mathbb{Z}}$ defined by $f(x) = 2x+8\mathbb{Z}$ induces a function $\tilde{f} : \dfrac{\mathbb{Z}}{4\mathbb{Z}} \to \dfrac{\mathbb{Z}}{8\mathbb{Z}}$ defined by

$$\tilde{f}(x + 4\mathbb{Z}) = 2x + 8\mathbb{Z}.$$

First,

$$f(x+y) = 2(x+y) + 8\mathbb{Z} = 2x + 2y + 8\mathbb{Z} = (2x + 8\mathbb{Z}) + (2y + 8\mathbb{Z}) = f(x) + f(y).$$

Hence, f is a group map.
Next, if $4n \in 4\mathbb{Z}$, then

$$f(4n) = 2(4n) + 8\mathbb{Z} = 8n + 8\mathbb{Z} = 8\mathbb{Z}.$$

By the Universal Property of the Quotient, f induces a function $\tilde{f} : \dfrac{\mathbb{Z}}{4\mathbb{Z}} \to \dfrac{\mathbb{Z}}{8\mathbb{Z}}$ defined by

$$\tilde{f}(x + 4\mathbb{Z}) = 2x + 8\mathbb{Z}. \quad \square$$

3. Show that the function $f : \mathbb{Z} \to \dfrac{\mathbb{Z}}{8\mathbb{Z}}$ defined by $f(x) = 3x + 8\mathbb{Z}$ does not induce a function $\tilde{f} : \dfrac{\mathbb{Z}}{4\mathbb{Z}} \to \dfrac{\mathbb{Z}}{8\mathbb{Z}}$ defined by

$$\tilde{f}(x + 4\mathbb{Z}) = 3x + 8\mathbb{Z}.$$

Specifically, show that f is a group map, but $\tilde{f}$ is not.

First,

$$f(x + y) = 3(x + y) + 8\mathbb{Z} = 3x + 3y + 8\mathbb{Z} = (3x + 8\mathbb{Z}) + (3y + 8\mathbb{Z}) = f(x) + f(y).$$

Hence, f is a group map.
However,

$$\tilde{f}[(1 + 4\mathbb{Z}) + (3 + 4\mathbb{Z})] = \tilde{f}(4 + 4\mathbb{Z}) = \tilde{f}(0 + 4\mathbb{Z}) = 3 \cdot 0 + 8\mathbb{Z} = 8\mathbb{Z},$$

$$\tilde{f}(1 + 4\mathbb{Z}) + \tilde{f}(3 + 4\mathbb{Z}) = (3 + 8\mathbb{Z}) + (9 + 8\mathbb{Z}) = 12 + 8\mathbb{Z} = 4 + 8\mathbb{Z}.$$

Thus, $\tilde{f}[(1 + 4\mathbb{Z}) + (3 + 4\mathbb{Z})] \neq \tilde{f}(1 + 4\mathbb{Z}) + \tilde{f}(3 + 4\mathbb{Z})$, so $\tilde{f}$ is not a group map. □

What lies in our power to do, it lies in our power not to do. - Aristotle