Math 345/504
4–13–2018

Solutions to Problem Set 24

1. Consider the ring $\mathbb{Z}_{10}$.

(a) List the zero divisors in $\mathbb{Z}_{10}$. For each zero divisor r, find a nonzero element s such that $rs = 0$.

(b) List the units in $\mathbb{Z}_{10}$. For each unit r, find the multiplicative inverse r^{-1}.

(a) I'll call a nonzero element which multiplies a zero divisor to 0 a *zero divisor complement*.

zero divisor	zero divisor complement
2	5
4	5
5	2
6	5
8	5

□

(b)

unit	inverse
1	1
3	7
7	3
9	9

□

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2. $\mathbb{Z}_2 \times \mathbb{Z}_6$ is a ring under componentwise addition and multiplication.

(a) List all the units in $\mathbb{Z}_2 \times \mathbb{Z}_6$. For each unit, give its multiplicative inverse.

(b) List all the zero divisors in $\mathbb{Z}_2 \times \mathbb{Z}_6$. For each zero divisor, give a nonzero element whose product with the zero divisor is the zero element.

(a)

unit	inverse
$(1, 1)$	$(1, 1)$
$(1, 5)$	$(1, 5)$

□

(b) For each zero divisor, I'll refer to a "nonzero element whose product with the zero divisor is the zero element" as the "complement".

zero divisor	complement
$(0,1)$	$(1,0)$
$(0,2)$	$(1,0)$
$(0,3)$	$(1,0)$
$(0,4)$	$(1,0)$
$(0,5)$	$(1,0)$
$(1,0)$	$(0,1)$
$(1,2)$	$(0,3)$
$(1,3)$	$(0,2)$
$(1,4)$	$(0,3)$

□

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3. Consider the following subsets of the ring $\mathbb{Z}\times\mathbb{Z}$. Check each axiom for an ideal. If the axiom holds, prove it. If the axiom does not hold, give a specific numerical counterexample.

(a) $8\mathbb{Z}\times 11\mathbb{Z}$.

$8\mathbb{Z}\times 11\mathbb{Z}$ consists of elements $(8a, 11b)$ in $\mathbb{Z}\times\mathbb{Z}$. For example $(16, 55)\in 8\mathbb{Z}\times 11\mathbb{Z}$.

(b) The set $S = \{(a,b) \mid a+b=0\}$.

(c) The set $T = \{(a,b) \mid ab=0\}$.

(a) $8\mathbb{Z}\times 11\mathbb{Z}$ is closed under addition, since

$$(8a, 11b) + (8c + 11d) = (8(a+c), 11(b+d)) \in 8\mathbb{Z}\times 11\mathbb{Z}.$$

Next, $(0,0) = (8\cdot 0, 11\cdot 0) \in 8\mathbb{Z}\times 11\mathbb{Z}$.
$8\mathbb{Z}\times 11\mathbb{Z}$ is closed under taking inverses, since

$$-(8a, 11b) = (8(-a), 11(-b)) \in 8\mathbb{Z}\times 11\mathbb{Z}.$$

Finally, if $(x,y)\in\mathbb{Z}\times\mathbb{Z}$,

$$(x,y)\cdot(8a, 11b) = (8(xa), 11(yb)) \in 8\mathbb{Z}\times 11\mathbb{Z}.$$

Multiplication in $\mathbb{Z}\times\mathbb{Z}$ is commutative, so this completes the proof that $8\mathbb{Z}\times 11\mathbb{Z}$ is an ideal. □

(b) Since $0+0=0$, it follows that $(0,0)\in S$.
If $(a,b)\in S$, then $a+b=0$. Hence, $(-a)+(-b)=0$. Therefore,

$$-(a,b) = (-a,-b) \in S.$$

If $(a,b), (c,d) \in S$, then

$$a+b=0, \quad c+d=0, \quad \text{so} \quad a+b+c+d=0.$$

Therefore,

$$(a,b)+(c,d) = (a+c, b+d) \in S.$$

$(1, -1) \in S$, since $1 + (-1) = 0$. But

$$(1, 2) \cdot (1, -1) = (1, -2) \notin S, \quad \text{since} \quad 1 + (-2) = -1 \neq 0.$$

Since S is not closed under multiplication by arbitrary ring elements, S is not an ideal. □

(c) T can be described as the set of pairs which have at least one zero component.
$(0, 0) \in T$ because $0 \cdot 0 = 0$.
If $(a, b) \in T$, then $ab = 0$. Hence, $(-a)(-b) = ab = 0$. Therefore,

$$-(a, b) = (-a, -b) \in T.$$

If $(a, b) \in T$, so $ab = 0$, and $(r, s) \in \mathbb{Z} \times \mathbb{Z}$, then

$$(r, s)(a, b) = (ra, sb).$$

Now

$$(ra)(sb) = (rs)(ab) = (rs) \cdot 0 = 0.$$

Hence, $(r, s)(a, b) \in T$.
$(1, 0), (0, 1) \in T$, since $1 \cdot 0 = 0$ and $0 \cdot 1 = 0$. However,

$$(1, 0) + (0, 1) = (1, 1) \notin T, \quad \text{since} \quad 1 \cdot 1 = 1 \neq 0.$$

Therefore, T is not an ideal. □

[Math 504]

4. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a commutative ring with identity under componentwise addition and multiplication.

(a) Show that $(7, -5, 0)$ is a zero divisor.

(b) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ has 8 units. List them.

(a) $(7, -5, 0) \cdot (0, 0, 1) = (0, 0, 0)$. □

(b)

$$(1, 1, 1), \quad (1, 1, -1), \quad (1, -1, 1), \quad (1, -1, -1),$$
$$(-1, 1, 1), \quad (-1, 1, -1), \quad (-1, -1, 1), \quad (-1, -1, -1).$$ □

If a man is called to be a streetsweeper, he should sweep streets even as Michelangelo painted, or Beethoven composed music or Shakespeare wrote poetry. He should sweep streets to well that all the hosts of heaven and earth will pause to say, here lived a great streetsweeper who did his job well. - Martin Luther King, Jr.