Math 345/504
4–23–2018

Solutions to Problem Set 27

1. Let R be an integral domain. Let

$$Q = \left\{(a, b) \;\middle|\; a, b \in R \quad \text{and} \quad b \neq 0\right\}.$$

Define a relation $\sim$ on Q by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

Prove that $\sim$ is an equivalence relation (that is, $\sim$ is reflexive, symmetric, and transitive).

$(a, b) \sim (a, b)$, because $ab = ba$. The relation is reflexive.
If $(a, b) \sim (c, d)$, then $ad = bc$, so $cb = da$, and hence $(c, d) \sim (a, b)$. The relation is symmetric.
If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ad = bc$ and $cf = de$. Since $d \neq 0$,

$$adf = bcf = bde, \quad \text{and} \quad af = be.$$

Therefore, $(a, b) \sim (e, f)$. The relation is transitive. $\square$

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2. $\mathbb{Z}_7(x)$ is the quotient field of $\mathbb{Z}_7[x]$. Elements of the quotient field may be represented by rational functions $\dfrac{f(x)}{g(x)}$, where $f(x), g(x) \in \mathbb{Z}_7[x]$.

Determine whether the following elements of the quotient field $\mathbb{Z}_7(x)$ are equal.

(a) $\dfrac{3x+2}{5x+3}$ and $\dfrac{2x+1}{x+5}$.

(b) $\dfrac{x+2}{4x+1}$ and $\dfrac{2x+1}{x+4}$.

(a)

$$(3x+2)(x+5) = 3x^2+3x+3 \quad \text{and} \quad (5x+3)(2x+1) = 3x^2+4x+3.$$

Hence, $\dfrac{3x+2}{5x+3} \neq \dfrac{2x+1}{x+5}$. $\square$

(b)

$$(x+2)(x+4) = x^2+6x+1 \quad \text{and} \quad (4x+1)(2x+1) = x^2+6x+1.$$

Hence, $\dfrac{x+2}{4x+1} = \dfrac{2x+1}{x+4}$. $\square$

[Math 504]

3. Let R be an integral domain. Let

$$Q = \left\{(a, b) \;\middle|\; a, b \in R \quad \text{and} \quad b \neq 0\right\}.$$

Define a relation $\sim$ on Q by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Prove that

$$(a_1 d_1 + b_1 c_1, b_1 d_1) \sim (a_2 d_2 + b_2 c_2, b_2 d_2) \quad \text{and} \quad (a_1 c_1, b_1 d_1) \sim (a_2 c_2, b_2 d_2).$$

Since $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$, I have $a_1 b_2 = b_1 a_2$ and $c_1 d_2 = d_1 c_2$. Now

$$(a_1 d_1 + b_1 c_1) b_2 d_2 = a_1 b_2 d_1 d_2 + b_1 b_2 c_1 d_2 = b_1 a_2 d_1 d_2 + b_1 b_2 d_1 c_2 = (a_2 d_2 + b_2 c_2) b_1 d_1.$$

This proves that $(a_1 d_1 + b_1 c_1, b_1 d_1) \sim (a_2 d_2 + b_2 c_2, b_2 d_2)$.
Also,

$$a_1 c_1 b_2 d_2 = b_1 a_2 d_1 c_2 = a_2 c_2 b_1 d_1.$$

This proves that $(a_1 c_1, b_1 d_1) \sim (a_2 c_2, b_2 d_2)$. $\square$

To do just the opposite is also a form of imitation. - G.-C. Lichtenberg

