Math 345/504
1–28–2018

Problem Set 4

Problem Set 4 is due by the end of class on Friday, February 2, or using a free makeup, by the end of class on Monday, February 5.

Problems marked "MATH 504" should be done only by MATH 504 students. MATH 345 students will not get credit for doing them. The other problems should be done by everyone.

1. In each case, a group and a subset of the group are given. Check EACH axiom for a subgroup. If the axiom holds, prove it; if the axiom doesn't hold, give a specific counterexample.

(a) In the group $GL(2,\mathbb{R})$ of 2×2 invertible matrices with real entries under matrix multiplication, consider the subset

$$H = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R}, x \neq 0 \right\}.$$

(b) In the group $\mathbb{R}^+$ of positive real numbers under multiplication, consider the subset

$$K = \left\{ (\sqrt{3})^n \mid n \in \mathbb{Z} \right\}.$$

(c) In the group $\mathbb{R}^+$ of positive real numbers under multiplication, consider the subset of positive integers

$$L = \{1, 2, 3, 4, \ldots\}.$$

(d) In the group $\mathbb{R}^2$ of pairs of real numbers under componentwise (vector) addition, consider the subset

$$P = \{(x, \pm x) \mid x \in \mathbb{R}\}.$$

(So, for example, $(5, 5)$ and $(9, -9)$ are elements of P.)

2. (a) Define $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ by $f(x) = 17x$. Prove that f is a group map.

(b) Define $g : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ by $g(x) = x + 13$. Prove that g is *not* a group map.

Note: In (b), you must give a specific counterexample.

3. Let G be a group, and suppose that $x^2 = 1$ for all $x \in G$. Prove that G is abelian.

Note: Let $x, y \in G$. You must show $xy = yx$. Use the fact that if $a^2 = 1$, then $a = a^{-1}$.

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4. Let G be a group. Define $f : G \to G$ by

$$f(x) = x^{-1}.$$

Prove that f is a group map if and only if G is abelian.

Note: Be sure that you prove *both* implications.