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The Column Space of a Matrix

Definition. Let A be an $m \times n$ matrix. The **column vectors** of A are the vectors in F^n corresponding to the columns of A. The **column space** of A is the subspace of F^n spanned by the column vectors of A.

For example, consider the real matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The column vectors are $(1,0,0)$ and $(0,1,0)$. The column space is the subspace of $\mathbb{R}^3$ spanned by these vectors. Thus, the column space consists of all vectors of the form

$$a \cdot (1,0,0) + b \cdot (0,1,0) = (a,b,0).$$

We've seen how to find a basis for the row space of a matrix. We'll now give an algorithm for finding a basis for the column space.

First, here's a reminder about matrix multiplication. If A is an $m \times n$ matrix and $v \in F^n$, then you can think of the multiplication Av as multiplying the columns of A by the components of v:

$$\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ c_1 & c_2 & \cdots & c_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \end{array} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

This means that if c_i is the i-th column of A and $v = (a_1, \ldots, a_n)$, the product Av is a linear combination of the columns of A:

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ c_1 & c_2 & \cdots & c_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.$$

Proposition. Let A be a matrix, and let R be the row reduced echelon matrix which is row equivalent to A. Suppose the leading entries of R occur in columns $j_1, \ldots, j_p$, where $j_1 < \cdots < j_p$, and let c_i denote the i-th column of A. Then $\{c_{j_1}, \ldots, c_{j_p}\}$ is independent.

Proof. Suppose that

$$a_{j_1} c_{j_1} + \cdots + a_{j_p} c_{j_p} = 0, \quad \text{for} \quad a_i \in F.$$

Form the vector $v = (v_i)$, where

$$v_i = \begin{cases} 0 & \text{if } i \notin \{j_1, \ldots, j_p\} \\ a_i & \text{if } i \in \{j_1, \ldots, j_p\} \end{cases}$$

The equation above implies that $Av = 0$.

It follows that v is in the solution space of the system $Ax = 0$. Since $Rx = 0$ has the same solution space, $Rv = 0$. Let c_i' denote the i-th column of R. Then

$$0 = Rv = a_{j_1} c_{j_1}' + \cdots + a_{j_p} c_{j_p}.$$

However, since R is in row reduced echelon form, c'_{j_k} is a vector with 1 in the k-th row and 0's elsewhere. Hence, $\{c_{j_1}, \ldots, c_{j_p}\}$ is independent, and $a_{j_1} = \cdots = a_{j_p} = 0$. $\square$

The proof provides an algorithm for finding a basis for the column space of a matrix. Specifically, row reduce the matrix A to a row reduced echelon matrix R. If the leading entries of R occur in columns $j_1, \ldots, j_p$, then consider the columns $c_{j_1}, \ldots, c_{j_p}$ of A. These columns form a basis for the column space of A. $\square$

Example. Find a basis for the column space of the real matrix

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 0 & -5 & 6 & -1 & 1 \\ 7 & 1 & 3 & 10 & 4 \end{bmatrix}.$$

Row reduce the matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 0 & -5 & 6 & -1 & 1 \\ 7 & 1 & 3 & 10 & 4 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0.6 & 1.4 & 0.6 \\ 0 & 1 & -1.2 & 0.2 & -0.2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading entries occur in columns 1 and 2. Therefore, $(1, 2, 0, 7)$ and $(-2, 1, -5, 1)$ form a basis for the column space of A. $\square$

Note that if A and B are row equivalent, they don't necessarily have the same column space. For example,

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\to} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, all the elements of the column space of the second matrix have their second component equal to 0; this is obviously not true of elements of the column space of the first matrix.

Example. Find a basis for the column space of the following matrix over $\mathbb{Z}_3$:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

Row reduce the matrix:

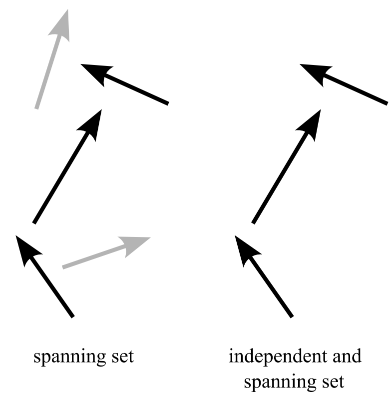
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\to} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix} \underset{r_3 \to r_3 + r_1}{\to}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\to} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The leading entries occur in columns 1, 2, and 4. Hence, columns 1, 2, and 4 of A are independent and form a basis for the column space of A:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \square$$

I showed earlier that you can *add vectors to an independent set to get a basis.* The column space basis algorithm shows how to *remove vectors from a spanning set to get a basis.*



Example. Find a subset of the following set of vectors which forms a basis for $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Make a matrix with the vectors as *columns* and row reduce:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 1 & -1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & \dfrac{2}{3} & 0 \\ 0 & 1 & -\dfrac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The leading entries occur in columns 1, 2, and 4. Therefore, the corresponding columns of the original matrix are independent, and form a basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}. \quad \square$$

Definition. Let A be a matrix. The **column rank** of A is the dimension of the column space of A.

This is really just a temporary definition, since we'll show that the column rank is the same as the rank we define earlier (the dimension of the row space).

Proposition. Let A be a matrix. Then

$$\operatorname{rank}(A) = \text{column rank}(A).$$

Proof. Let R be the row reduced echelon matrix which is row equivalent to A. Suppose the leading entries of R occur in columns $j_1, \ldots, j_p$, where $j_1 < \cdots < j_p$, and let c_i denote the i-th column of A. By the preceding lemma, $\{c_{j_1}, \ldots, c_{j_p}\}$ is independent. There is one vector in this set for each leading entry, and the number of leading entries equals the row rank. Therefore,

$$\operatorname{rank}(A) \le \text{column rank}(A).$$

Now consider A^T. This is A with the rows and columns swapped, so

$$\operatorname{rank}(A^T) = \text{column rank}(A),$$

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Applying the first part of the proof to A^T,

$$\text{column rank}(A) = \operatorname{rank}(A^T) \le \text{column rank}(A^T) = \operatorname{rank}(A).$$

Therefore,

$$\text{column rank}(A) = \operatorname{rank}(A). \quad \square$$

Proposition. Let A, B, P and Q be matrices, where P and Q are invertible. Suppose $A = PBQ$. Then

$$\operatorname{rank} A = \operatorname{rank} B.$$

Proof. I showed earlier that $\operatorname{rank} MN \le \operatorname{rank} N$. This was row rank; a similar proof shows that

$$\text{column rank}(MN) \le \text{column rank}(M).$$

Since row rank and column rank are the same, $\operatorname{rank} MN \le \operatorname{rank} M$.
Now

$$\operatorname{rank} A = \operatorname{rank} PBQ \le \operatorname{rank} BQ = \text{column rank}(BQ) \le \text{column rank}(B) = \operatorname{rank} B.$$

But $B = P^{-1}AQ^{-1}$, so repeating the computation gives $\operatorname{rank} B \le \operatorname{rank} A$. Therefore, $\operatorname{rank} A = \operatorname{rank} B$.
$\square$

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