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Complex Numbers

A **complex number** is a number of the form $a + bi$, where a and b are real numbers. a is called the **real part** and b is called the **imaginary part**; the notation is

$$\mathrm{re}(a + bi) = a, \quad \mathrm{im}(a + bi) = b.$$

You add, subtract, and multiply complex numbers in the obvious ways:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The **conjugate** of complex number is obtained by flipping the sign of the imaginary part. The conjugate of $a + bi$ is denoted $\overline{a + bi}$ or sometimes $(a + bi)^*$. Thus,

$$\overline{a + bi} = a - bi.$$

You can divide by (nonzero) complex numbers by "multiplying the top and bottom by the conjugate":

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}.$$

With these operations, the set of complex numbers forms a **field**.

Example.

$$2(4 - 5i) + (3 - 2i) = (8 - 10i) + (3 - 2i) = 11 - 12i.$$

$$(7 + 2i)(5 - 4i) = 35 + 10i - 28i - 8i^2 = 35 - 18i - 8(-1) = 43 - 18i.$$

$$\frac{5 - 2i}{3 + 4i} = \frac{5 - 2i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{15 - 6i - 20i + 8i^2}{9 - 16i^2} = \frac{15 - 26i + 8(-1)}{9 - 16(-1)} = \frac{7 - 26i}{25}.$$

$$i^{517} = i^{516} \cdot i = (i^2)^{258} \cdot i = (-1)^{258} \cdot i = 1 \cdot i = i. \quad \square$$

The **norm** of a complex number is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Note that

$$(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2.$$

When a complex number is written in the form $a + bi$, it's said to be in **rectangular form**. There is another form for complex numbers that is useful: The polar form $re^{i\theta}$. In this form, r and θ have the same meanings that they do in polar coordinates.

DeMoivre's formula relates the polar and rectangular forms:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

This key result can be proven, for example, by expanding both sides in power series. It's really useful, as you'll see in the examples below.

Observe that

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1.$$

Thus, $e^{i\theta}$ is a complex number of norm 1.

Example. Convert $3 + 4i$ to polar form.

$$3 + 4i = |3 + 4i| \cdot \frac{3 + 4i}{|3 + 4i|} = 5\left(\frac{3}{5} + \frac{4}{5}i\right).$$

Let $\theta = \sin^{-1}\dfrac{4}{5}$ (or $\cos^{-1}\dfrac{3}{5}$). Then

$$3 + 4i = 5(\cos\theta + i\sin\theta). \quad \square$$

Example. (**A trick with Demoivre's formula**) Find $(1 + i\sqrt{3})^8$.

It would be tedious to try to multiply this out. Instead, I'll try to write the expression in terms of $\cos\theta + i\sin\theta$ for a good choice of θ.

$$(1 + i\sqrt{3})^8 = 2^8 \cdot \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^8 = 256\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8 = 256(e^{\pi i/3})^8 = 256e^{8\pi i/3} =$$

$$256\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right) = 256\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -128 + 128i\sqrt{3}. \quad \square$$

Example. (**Proving trig identities**) Prove the **angle addition formulas**:

$$\cos(a + b) = \cos a\cos b - \sin a\sin b.$$

$$\sin(a + b) = \sin a\cos b + \sin b\cos a.$$

I have

$$\begin{aligned}
e^{(a+b)i} &= e^{ai}e^{bi} \\
\cos(a + b) + i\sin(a + b) &= (\cos a + i\sin a)(\cos b + i\sin b) \\
\cos(a + b) + i\sin(a + b) &= (\cos a\cos b - \sin a\sin b) + i(\sin a\cos b + \sin b\cos a)
\end{aligned}$$

Equating real and imaginary parts on the left and right sides, I get

$$\cos(a + b) = \cos a\cos b - \sin a\sin b \quad \text{(real parts)}.$$

$$\sin(a + b) = \sin a\cos b + \sin b\cos a \quad \text{(imaginary parts)}. \quad \square$$

Example. (**Computing integrals**) Compute $\displaystyle\int e^{2x}\cos 3x\,dx$.

Note that $\cos 3x = \operatorname{re}(e^{3xi})$. Thus,

$$\int e^{2x}\cos 3x\,dx = \int e^{2x}\operatorname{re}(e^{3xi})\,dx = \operatorname{re}\int e^{2x}e^{3xi}\,dx = \operatorname{re}\int e^{(2+3i)x}\,dx = \operatorname{re}\frac{1}{2+3i}e^{(2+3i)x} + c =$$

$$\operatorname{re}\frac{2-3i}{13}e^{2x}(\cos 3x + i\sin 3x) + c = \frac{1}{13}e^{2x}\operatorname{re}(2-3i)(\cos 3x + i\sin 3x) + c =$$

$$\frac{1}{13}e^{2x}\operatorname{re}\left((2\cos 3x + 3\sin 3x) + i(2\sin 3x - 3\cos 3x)\right) + c = \frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x) + c. \quad \square$$