Functions

Definition. A function $f$ from a set $X$ to a set $Y$ is a subset $S$ of the product $X \times Y$ such that if $(x, y_1), (x, y_2) \in S$, then $y_1 = y_2$.

Instead of writing $(x, y) \in S$, you usually write $f(x) = y$. Thus, when an ordered pair $(x, y)$ is in $S$, $x$ is the input to $f$ and $y$ is the corresponding output. The stipulation that

“If $(x, y_1), (x, y_2) \in S$, then $y_1 = y_2$”

means that there is a unique output for each input.

$f : X \to Y$ means that $f$ is a function from $X$ to $Y$. $X$ is called the domain, and $Y$ is called the codomain. The range of $f$ is the set of all outputs of the function:

$\{ y \in Y \mid y = f(x) \text{ for some } x \in X \}$.

Example. Consider the following functions:

$f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

$g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by $g(x) = x^2$.

$h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $h(x) = x^2$.

These are different functions; they’re defined by the same rule, but they have different domains or codomains.

Example. Suppose I try to define $f : \mathbb{R} \to \mathbb{R}$ by

$f(x) = (\text{an integer bigger than } x)$.

This does not define a function. For example, $f(2.6)$ could be 3, since 3 is an integer bigger than 2.6. But it could also be 4, or 67, or 101, or . . . . This “rule” does not produce a unique output for each input.

Mathematicians say that such a function — or such an attempted function — is not well-defined.

Example. (The “natural domain” of a function) In basic algebra and calculus, functions are often given by rules, without mention of a domain and codomain; for example,

$f(x) = \frac{\sqrt{x}}{x - 2}$.

In this context, the codomain is usually understood to be $\mathbb{R}$. What about the domain? In this situation, the domain is understood to be the largest subset of $\mathbb{R}$ for which $f$ is defined. (This is sometimes called the natural domain of $f$.) In the case of $f(x) = \frac{\sqrt{x}}{x - 2}$:

- I must have $x \geq 0$ in order for $\sqrt{x}$ to be defined.
- I must have $x \neq 2$ in order to avoid division by zero.

Hence, the domain of $f$ is $[0, 2) \cup (2, +\infty)$. □
**Definition.** Let \( X \) and \( Y \) be sets. A function \( f : X \rightarrow Y \) is:

(a) **Injective** if for all \( x_1, x_2 \in X \), \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \).

(b) **Surjective** if for all \( y \in Y \), there is an \( x \in X \) such that \( f(x) = y \).

(c) **Bijective** if it is injective and surjective.

**Definition.** Let \( S \) and \( T \) be sets, and let \( f : S \rightarrow T \) be a function from \( S \) to \( T \). A function \( g : T \rightarrow S \) is called the **inverse** of \( f \) if

\[
g(f(s)) = s \quad \text{for all} \quad s \in S \quad \text{and} \quad f(g(t)) = t \quad \text{for all} \quad t \in T.
\]

Not all functions have inverses; if the inverse of \( f \) exists, it’s denoted \( f^{-1} \). *(Despite the crummy notation, “\( f^{-1} \)” does not mean “\( 1/f \)”)*.

You’ve undoubtedly seen inverses of functions in other courses; for example, the inverse of \( f(x) = x^3 \) is \( f^{-1}(x) = x^{1/3} \). However, the functions I’m discussing may not have anything to do with numbers, and may not be defined using formulas.

**Example.** Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = \frac{x}{x + 1} \). To find the inverse of \( f \) (if there is one), set \( y = \frac{x}{x + 1} \). Swap the \( x \)'s and \( y \)'s, then solve for \( y \) in terms of \( x \):

\[
x = \frac{y}{y + 1}, \quad x(y + 1) = y, \quad xy + x = y, \quad x = y - xy, \quad x = y(1 - x), \quad y = \frac{x}{1 - x}.
\]

Thus, \( f^{-1}(x) = \frac{x}{1 - x} \). To prove that this works using the definition of an inverse function, do this:

\[
f^{-1}(f(x)) = f^{-1}\left(\frac{x}{x + 1}\right) = \frac{x}{\frac{1 - x}{x + 1}} = \frac{x}{x + 1 - x} = \frac{x}{1} = x,
\]

\[
f\left(f^{-1}(x)\right) = f\left(\frac{x}{1 - x}\right) = \frac{x}{\frac{1 - x}{x + 1}} = \frac{x}{x + (1 - x)} = \frac{x}{1} = x.
\]

Recall that the graphs of \( f \) and \( f^{-1} \) are mirror images across the line \( y = x \):
I'm mentioning this to connect this discussion to things you've already learned. However, you should not make the mistake of equating this special case with the definition. The inverse of a function is not defined by "swapping x's and y's and solving" or "reflecting the graph about y = x". A function might not involve numbers or formulas, and a function might not have a graph. The inverse of a function is what the definition says it is — nothing more or less. □

The next lemma is very important, and I'll often use it in showing that a given function is bijective.

**Lemma.** Let $S$ and $T$ be sets, and let $f : S \to T$ be a function. $f$ is invertible if and only if $f$ is both injective and surjective.

**Proof.** $(\to)$ Suppose that $f$ is both injective and surjective. I'll construct the inverse function $f^{-1} : T \to S$.

Take $t \in T$. Since $f$ is surjective, there is an element $s \in S$ such that $f(s) = t$. Moreover, $s$ is unique: if $f(s) = t$ and $f(s') = t$, then $f(s) = f(s')$. But $f$ is injective, so $s = s'$.

Define

$$f^{-1}(t) = s.$$ 

I have defined a function $f^{-1} : T \to S$. I must show that it is the inverse of $f$.

Let $s \in S$. By definition of $f^{-1}$, to compute $f^{-1}(f(s))$ I must find an element $Moe \in S$ such that $f(Moe) = f(s)$. But this is easy — just take $Moe = s$. Thus, $f^{-1}(f(s)) = s$.

Going the other way, let $t \in T$. By definition of $f^{-1}$, to compute $f(f^{-1}(t))$ I find an element $s \in S$ such that $f(s) = t$. Then $f^{-1}(t) = s$, so

$$f(f^{-1}(t)) = f(s) = t.$$ 

Therefore, $f^{-1}$ really is the inverse of $f$.

$(\leftarrow)$ Suppose $f$ has an inverse $f^{-1} : T \to S$. I must show $f$ is injective and surjective.

To show that $f$ is surjective, take $t \in T$. Then $f(f^{-1}(t)) = t$, so I've found an element of $S$ — namely $f^{-1}(t)$ — which $f$ maps to $t$. Therefore, $f$ is surjective.

To show that $f$ is injective, suppose $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$. Then

$$f^{-1}(f(s_1)) = f^{-1}(f(s_2)), \quad \text{so} \quad s_1 = s_2.$$ 

Therefore, $f$ is injective. □

**Example.** (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither injective nor surjective.

It is not injective, since $f(-3) = 9$ and $f(3) = 9$: Different inputs may give the same output.

It is not surjective, since there is no $x \in \mathbb{R}$ such that $f(x) = -1$. □

(b) $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$ is not injective, but it is surjective.

It is not injective, since $f(-3) = 9$ and $f(3) = 9$: Different inputs may give the same output.

It is surjective, since if $y \geq 0$, $\sqrt{y}$ is defined, and

$$f(\sqrt{y}) = (\sqrt{y})^2 = y.$$ 

(c) $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$ is injective and surjective.

It is injective, since if $f(x_1) = f(x_2)$, then $x_1^2 = x_2^2$. But in this case, $x_1, x_2 \geq 0$, so $x_1 = x_2$ by taking square roots.

It is surjective, since if $y \geq 0$, $\sqrt{y}$ is defined, and

$$f(\sqrt{y}) = (\sqrt{y})^2 = y.$$ 

□
Notice that in this example, the same “rule” — $f(x) = x^2$ — was used, but whether the function was injective or surjective changed. The domain and codomain are part of the definition of a function.

**Example.** Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x+1}{x}$ is injective.

Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$. I must prove that $x_1 = x_2$.

$f(x_1) = f(x_2)$ means that $\frac{x_1+1}{x_1} = \frac{x_2+1}{x_2}$. Clearing denominators and doing some algebra, I get

$$x_2(x_1+1) = x_1(x_2+1), \quad x_2x_1 + x_2 = x_1x_2 + x_1, \quad x_1 = x_2.$$ Therefore, $f$ is injective.

**Example.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x - 7$.

(a) Prove directly that $f$ is injective and surjective.

Suppose $f(a) = f(b)$. Then $5a - 7 = 5b - 7$, so $5a = 5b$, and hence $a = b$. Therefore, $f$ is injective.

Suppose $y \in \mathbb{R}$. I must find $x$ such that $f(x) = y$. I want $5x - 7 = y$. Working backwards, I find that $x = \frac{1}{5}(y + 7)$. Verify that it works:

$$f\left(\frac{1}{5}(y + 7)\right) = 5 \cdot \frac{1}{5}(y + 7) - 7 = (y + 7) - 7 = y.$$ This proves that $f$ is surjective.

(b) Prove that $f$ is injective and surjective by showing that $f$ has an inverse $f^{-1}$.

Define $f^{-1}(x) = \frac{1}{5}(x + 7)$. I’ll prove that this is the inverse of $f$:

$$f(f^{-1}(x)) = f\left(\frac{1}{5}(x + 7)\right) = 5 \cdot \frac{1}{5}(x + 7) - 7 = (x + 7) - 7 = x,$$

$$f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{1}{5}((5x - 7) + 7) = \frac{1}{5} \cdot 5x = x.$$ Therefore, $f^{-1}$ is the inverse of $f$. Since $f$ is invertible, it’s injective and surjective.

**Example.** In some cases, it may be difficult to prove that a function is surjective by a direct argument.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2(x - 1)$. $f$ is not injective, since $f(0) = 0$ and $f(1) = 0$.

The graph suggests that $f$ is surjective. To say that every $y \in \mathbb{R}$ is an output of $f$ means graphically that every horizontal line crosses the graph at least once (whereas injectivity means that every horizontal line crosses that graph at most once).
To prove that \( f \) is surjective, take \( y \in \mathbb{R} \). I must find \( x \in \mathbb{R} \) such that \( f(x) = y \), i.e. such that 
\[
 x^2(x - 1) = y.
\]
The problem is that finding \( x \) in terms of \( y \) involves solving a cubic equation. This is possible, but it’s easy to change the example to produce a function where solving algebraically is impossible in principle.

Instead, I’ll proceed indirectly.\[
\lim_{x \to +\infty} x^2(x - 1) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} x^2(x - 1) = -\infty.
\]

It follows from the definition of these infinite limits that there are numbers \( x_1 \) and \( x_2 \) such that
\[
f(x_1) < y \quad \text{and} \quad f(x_2) > y.
\]

But \( f \) is continuous — it’s a polynomial — so by the Intermediate Value Theorem, there is a point \( c \) such that \( x_1 < c < x_2 \) and \( f(c) = y \). This proves that \( f \) is surjective.

Note, however, that I haven’t found \( c \); I’ve merely shown that such a value \( c \) must exist. ∎

**Example.** Define
\[
f(x) = \begin{cases} 
  x + 1 & \text{if } x < 0 \\
  x^2 & \text{if } x \geq 0
\end{cases}.
\]

Prove that \( f \) is surjective, but not injective.

Let \( y \in \mathbb{R} \). If \( y < 0 \), then \( y - 1 < -1 < 0 \), so
\[
f(y - 1) = (y - 1) + 1 = y.
\]

If \( y \geq 0 \), then \( \sqrt{y} \) is defined and \( \sqrt{y} \geq 0 \), so
\[
f(\sqrt{y}) = (\sqrt{y})^2 = y.
\]

This proves that \( f \) is surjective. However,
\[
f\left(-\frac{3}{4}\right) = -\frac{3}{4} + 1 = \frac{1}{4} \quad \text{and} \quad f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.
\]

Hence, \( f \) is not injective. ∎

**Example.** Consider the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
\[
f(x, y) = (4x - 3y, 2x + y).
\]
(a) Show that $f$ is injective and surjective directly, using the definitions.

First, I’ll show that $f$ is injective. Suppose $f(x_1, y_1) = f(x_2, y_2)$. I want to show that $(x_1, y_1) = (x_2, y_2)$. $f(x_1, y_1) = f(x_2, y_2)$ means

$$(4x_1 - 3y_1, 2x_1 + y_1) = (4x_2 - 3y_2, 2x_2 + y_2).$$

Equate corresponding components:

$$4x_1 - 3y_1 = 4x_2 - 3y_2, \quad 2x_1 + y_1 = 2x_2 + y_2.$$

Rewrite the equations:

$$4(x_1 - x_2) = 3(y_1 - y_2), \quad 2(x_1 - x_2) = -(y_1 - y_2).$$

The second of these equations gives $y_1 - y_2 = -2(x_1 - x_2)$. Substitute this into the first equation:

$$4(x_1 - x_2) = 3 \cdot (-2)(x_1 - x_2), \quad 4(x_1 - x_2) = -6(x_1 - x_2), \quad 10(x_1 - x_2) = 0, \quad x_1 - x_2 = 0, \quad x_1 = x_2.$$

Plugging this into $y_1 - y_2 = -2(x_1 - x_2)$ gives $y_1 - y_2 = 0$, so $y_1 = y_2$. Therefore, $(x_1, y_1) = (x_2, y_2)$, and $f$ is injective.

To show $f$ is surjective, I take a point $(a, b) \in \mathbb{R}^2$, the codomain. I must find $(x, y)$ such that $f(x, y) = (a, b)$.

I want

$$4(x - 3y, 2x + y) = (a, b).$$

I’ll work backwards from this equation. Equating corresponding components gives

$$4x - 3y = a, \quad 2x + y = b.$$

The second equation gives $y = b - 2x$, so plugging this into the first equation yields

$$4x - 3(b - 2x) = a, \quad 10x - 3b = a, \quad x = 0.1a + 0.3b.$$

Plugging this back into $y = b - 2x$ gives

$$y = b - 2(0.1a + 0.3b) = -0.2a + 0.4b.$$

Now check that this works:

$$f(0.1a + 0.3b, -0.2a + 0.4b) = (4(0.1a + 0.3b) - 3(-0.2a + 0.4b), 2(0.1a + 0.3b) + (-0.2a + 0.4b)) = (a, b).$$

Therefore, $f$ is surjective. □

(b) Show that $f$ is injective and surjective by constructing an inverse $f^{-1}$.

I actually did the work of constructing the inverse in showing that $f$ was surjective: I showed that if $f(x, y) = (a, b)$, that

$$(x, y) = (0.1a + 0.3b, -0.2a + 0.4b), \quad \text{or} \quad f(0.1a + 0.3b, -0.2a + 0.4b) = (a, b).$$

But the second equation implies that if $f^{-1}$ exists, it should be defined by

$$f^{-1}(a, b) = (0.1a + 0.3b, -0.2a + 0.4b).$$

Now I showed above that

$$f(f^{-1}(a, b)) = f(0.1a + 0.3b, -0.2a + 0.4b) = (a, b).$$
For the other direction,

\[ f^{-1}(f(x, y)) = f^{-1}(4x - 3y, 2x + y) = (0.1(4x - 3y) + 0.3(2x + y), -0.2(4x - 3y) + 0.4(2x + y)) = (x, y). \]

This proves that \( f^{-1} \), as defined above, really is the inverse of \( f \). Hence, \( f \) is injective and surjective.

\[ \square \]

**Remark.** In linear algebra, you learn more efficient ways to show that functions like the one above are bijective.

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**Example.** Consider the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[ f(x, y) = (2x + 4y, -x - 2y). \]

Prove that \( f \) is neither injective nor surjective.

\[ f(0, 0) = (0, 0) \quad \text{and} \quad f(2, -1) = (0, 0). \]

Therefore, \( f \) is not injective.

To prove \( f \) is not surjective, I must find a point \((a, b) \in \mathbb{R}^2\) which is not an output of \( f \). I’ll show that \((1, 1)\) is not an output of \( f \). Suppose on the contrary that \( f(x, y) = (1, 1) \). Then

\[ (2x + 4y, -x - 2y) = (1, 1). \]

This gives two equations:

\[ 2x + 4y = 1, \quad -x - 2y = 1. \]

Multiply the second equation by \(-2\) to obtain \(2x + 4y = -2\). Now I have \(2x + 4y = 1\) and \(2x + 4y = -2\), so \(1 = -2\), a contradiction.

Therefore, there is no such \((x, y)\), and \( f \) is not surjective. \( \square \)

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**Example.** Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by

\[ f(x, y) = (e^x + y, y^3). \]

Is \( f \) injective? Is \( f \) surjective?

First, I’ll show that \( f \) is injective. Suppose \( f(a, b) = f(c, d) \). I have to show that \((a, b) = (c, d)\).

\[ f(a, b) = f(c, d) \]

\[ (e^a + b, b^3) = (e^c + d, d^3) \]

Equating the second components, I get \( b^3 = d^3 \). By taking cube roots, I get \( b = d \). Equating the first components, I get \( e^a + b = e^c + d \). But \( b = d \), so subtracting \( b = d \) I get \( e^a = e^c \). Now taking the log of both sides gives \( a = c \). Thus, \((a, b) = (c, d)\), and \( f \) is injective.

I’ll show that \( f \) is not surjective by showing that there is no input \((x, y)\) which gives \(-1, 0\) as an output. Suppose on the contrary that \( f(x, y) = (-1, 0) \). Then

\[ f(x, y) = (-1, 0) \]

\[ (e^x + y, y^3) = (-1, 0) \]

Equating the second components gives \( y^3 = 0 \), so \( y = 0 \). Equating the first components gives \( e^x + y = -1 \). But \( y = 0 \), so I get \( e^x = -1 \). This is impossible, since \( e^x \) is always positive. Therefore, \( f \) is not surjective. \( \square \)
**Definition.** Let $A$, $B$, and $C$ be sets, and let $f : A \to B$ and $g : B \to C$ be functions. The **composite** of $f$ and $g$ is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(a) = g(f(a)).$$

I actually used composites earlier — for example, in defining the inverse of a function. In my opinion, the notation “$g \circ f$” looks a lot like multiplication, so (at least when elements are involved) I prefer to write “$g(f(x))$” instead. However, the composite notation is used often enough that you should be familiar with it.

**Example.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x + 1$. Then

$$(g \circ f)(x) = g(f(x)) = g(x^3) = x^3 + 1,$$

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^3.$$  

**Lemma.** Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Then $g \circ f$ is invertible, and its inverse is

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$  

**Proof.** Let $x \in X$ and let $z \in Z$. Then

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1} (g^{-1} (g (f(x)))) = f^{-1} (f(x)) = x,$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g \left( f \left( f^{-1} (g(z)) \right) \right) = g(g(z)) = z.$$  

This proves that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.  

**Corollary.** The composite of bijective functions is bijective.

**Proof.** I showed earlier that a function is bijective if and only if it has an inverse, so the corollary follows from the lemma.