Functions

Definition. A function $f$ from a set $X$ to a set $Y$ is a subset $f$ of the product $X \times Y$ such that if $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$.

Instead of writing $(x, y) \in f$, you usually write $f(x) = y$. In ordinary terms, to say that an ordered pair $(x, y)$ is in $f$ means that $x$ is the “input” to $f$ and $y$ is the corresponding “output”.

The requirement that $(x, y_1), (x, y_2) \in f$ implies $y_1 = y_2$ means that there is a unique output for each input. (It’s what is referred to as the “vertical line test” for a graph to be a function graph.)

(Why not say, as in precalculus or calculus classes, that a function is a rule that assigns a unique element of $Y$ to each element of $X$? The problem is that the word “rule” is ambiguous. The definition above avoids this by identifying a function with its graph.)

The notation $f : X \rightarrow Y$ means that $f$ is a function from $X$ to $Y$.

$X$ is called the domain, and $Y$ is called the codomain. The image (or range) of $f$ is the set of all outputs of the function:

$$\text{im } f = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}.$$

Note that the domain and codomain are part of the definition of a function. For example, consider the following functions:

- $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$.
- $g : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ given by $g(x) = x^2$.
- $h : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ given by $h(x) = x^2$.

These are different functions; they’re defined by the same rule, but they have different domains or codomains.

Example. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f(x) = (\text{an integer bigger than } x).$$

Is this a function?

This does not define a function. For example, $f(2.6)$ could be 3, since 3 is an integer bigger than 2.6. But it could also be 4, or 67, or 101, or . . . . This “rule” does not produce a unique output for each input.

Mathematicians say that such a function — or such an attempted function — is not well-defined.

In basic algebra and calculus, functions $\mathbb{R} \rightarrow \mathbb{R}$ are often given by rules, without mention of a domain and codomain. In this case, the natural domain (“domain” for short) is the largest subset of $\mathbb{R}$ consisting of numbers which can be “legally” plugged into the function.

Example. Find the natural domain of

$$f(x) = \frac{\sqrt{x}}{x - 2}.$$

(i) I must have $x \geq 0$ in order for $\sqrt{x}$ to be defined.

(ii) I must have $x \neq 2$ in order to avoid division by zero.

Hence, the domain of $f$ is $[0, 2) = (2, +\infty)$. □
**Example.** Define \( f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} \) by

\[
f(x) = \frac{3x}{x - 2}.
\]

Prove that \( \text{im } f = \{y \mid y \neq 3\} \).

In words, the claim is that the outputs of \( y \) consist of all numbers other than 3. To see why 3 might be omitted, note that

\[
\lim_{x \to \infty} \frac{3x}{x - 2} = 3.
\]

That is, \( y = 3 \) is a horizontal asymptote for the graph. Now this isn’t a proof, because a graph can cross a horizontal asymptote; it just provides us with a “guess”.

To prove \( \text{im } f = \{y \mid y \neq 3\} \), I’ll show each set is contained in the other.

Suppose \( y \neq 3 \). On scratch paper, I solve \( y = \frac{3x}{x - 2} \) for \( x \) in terms of \( y \) and get \( x = \frac{2y}{3 - y} \). (This is defined, since \( y \neq 3 \).) Now I prove that this input produces \( y \) as an output:

\[
f\left(\frac{2y}{3 - y}\right) = \frac{2y}{3 - y} \cdot \frac{2y}{3 - y} - \frac{2y}{3 - y} = \frac{6y}{2y - 2(3 - y)} = \frac{6y}{6} = y
\]

This proves that \( y \in \text{im } f \), so \( \{y \mid y \neq 3\} \subset \text{im } f \).

Conversely, suppose \( y \in \text{im } f \), so \( y = f(x) \) for some \( x \). I must show that \( y \neq 3 \). I’ll use proof by contradiction: Suppose \( y = 3 \). Then

\[
f(x) = y
\]

\[
\frac{3x}{x - 2} = 3
\]

\[
3x = 3(x - 2)
\]

\[
3x = 3x - 6
\]

\[
0 = -6
\]

This contradiction proves \( y \neq 3 \). Thus, \( \text{im } f \subset \{y \mid y \neq 3\} \). Therefore, \( \text{im } f = \{y \mid y \neq 3\} \). □

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**Definition.** Let \( X \) and \( Y \) be sets. A function \( f : X \rightarrow Y \) is:

(a) **Injective** if for all \( x_1, x_2 \in X \), \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \).

(b) **Surjective** if for all \( y \in Y \), there is an \( x \in X \) such that \( f(x) = y \).

(c) **Bijective** if it is injective and surjective.

**Definition.** Let \( A, B, \) and \( C \) be sets, and let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be functions. The **composite** of \( f \) and \( g \) is the function \( g \circ f : A \rightarrow C \) defined by

\[(g \circ f)(a) = g(f(a)).\]
In my opinion, the notation “$g \circ f$” looks a lot like multiplication, so (at least when elements are involved) I prefer to write “$g(f(x))$” instead. However, the composite notation is used often enough that you should be familiar with it.

**Example.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x + 1$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

$$(g \circ f)(x) = g(f(x)) = g(x^3) = x^3 + 1,$$

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^3.$$

**Example.** Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (x + y, x^2 + y) \quad \text{and} \quad g(x, y) = (y^3, x + y).$$

Find:

(a) $f(g(x, y))$.

(b) $f(f(x, y))$.

(a) $f(g(x, y)) = f(y^3, x + y) = (y^3 + (x + y), (y^3)^2 + (x + y)) = (x + y + y^3, x + y + y^6)$.

(b) $f(f(x, y)) = f(x + y, x^2 + y) = ((x + y) + (x^2 + y), (x + y)^2 + (x^2 + y)) = (x^2 + 2x + y, 2x^2 + 2xy + y^2 + y)$.

If you get confused doing this, keep in mind two things:

(i) The variables used in defining a function are “dummy variables” — just placeholders. For example, $f(a, b) = (a + b, a^2 + b)$ defines the same function $f$ as above.

(ii) The variables are “positional”, so in “$f(x, y)$” the “$x$” stands for “the first input to $f$” and the “$y$” stands for “the second input to $f$”. In fact, you might find it helpful to rewrite the definition of $f$ this way:

$$f((\text{first}), (\text{second})) = ((\text{first}) + (\text{second}), (\text{first})^2 + (\text{second})).$$

**Definition.** Let $S$ and $T$ be sets, and let $f : S \to T$ be a function from $S$ to $T$. A function $g : T \to S$ is called the inverse of $f$ if

$$g(f(s)) = s \quad \text{for all} \quad s \in S \quad \text{and} \quad f(g(t)) = t \quad \text{for all} \quad t \in T.$$

Not all functions have inverses; if the inverse of $f$ exists, it’s denoted $f^{-1}$. (Despite the crummy notation, “$f^{-1}$” does not mean “$\frac{1}{f}$”.)

You’ve undoubtedly seen inverses of functions in other courses; for example, the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. However, the functions I’m discussing may not have anything to do with numbers, and may not be defined using formulas.
Example. Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \frac{x}{x+1} \). Find the inverse of \( f \).

To find the inverse of \( f \) (if there is one), set \( y = \frac{x}{x+1} \). Swap the \( x \)'s and \( y \)'s, then solve for \( y \) in terms of \( x \):

\[
x = \frac{y}{y+1}, \quad x(y+1) = y, \quad xy + x = y, \quad x = y - xy, \quad x = y(1-x), \quad y = \frac{x}{1-x}.
\]

Thus, \( f^{-1}(x) = \frac{x}{1-x} \). To prove that this works using the definition of an inverse function, do this:

\[
f^{-1}(f(x)) = f^{-1}\left(\frac{x}{x+1}\right) = \frac{x}{1 - \frac{x}{x+1}} = \frac{x(x+1)}{(x+1)-x} = \frac{x}{1} = x,
\]

\[
f(f^{-1}(x)) = f\left(\frac{x}{1-x}\right) = \frac{x}{1-x + \frac{1}{1-x}} = \frac{x}{x + (1-x)} = \frac{x}{1} = x.
\]

Recall that the graphs of \( f \) and \( f^{-1} \) are mirror images across the line \( y = x \):

I'm mentioning this to connect this discussion to things you've already learned. However, you should not make the mistake of equating this special case with the definition. The inverse of a function is not defined by “swapping \( x \)'s and \( y \)'s and solving” or “reflecting the graph about \( y = x \).” A function might not involve numbers or formulas, and a function might not have a graph. The inverse of a function is what the definition says it is — nothing more or less.

Lemma. Let \( f : X \to Y \) and \( g : Y \to Z \) be invertible functions. Then \( g \circ f \) is invertible, and its inverse is

\[
(g \circ f)^{-1} = f^{-1} \circ g^{-1}.
\]

Proof. Let \( x \in X \) and let \( z \in Z \). Then

\[
(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x)))) = f^{-1}(f(x)) = x,
\]

\[
(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g(f(f^{-1}(g(z)))) = g(g(z)) = z.
\]

This proves that \( (g \circ f)^{-1} = f^{-1} \circ g^{-1} \).
**Theorem.** Let $S$ and $T$ be sets, and let $f : S \to T$ be a function. $f$ is invertible if and only if $f$ is both injective and surjective.

**Proof.** ($\to$) Suppose that $f$ is both injective and surjective. I’ll construct the inverse function $f^{-1} : T \to S$.

Take $t \in T$. Since $f$ is surjective, there is an element $s \in S$ such that $f(s) = t$. Moreover, $s$ is unique: if $f(s) = t$ and $f(s') = t$, then $f(s) = f(s')$. But $f$ is injective, so $s = s'$.

Define

$$f^{-1}(t) = s.$$ 

I have defined a function $f^{-1} : T \to S$. I must show that it is the inverse of $f$.

Let $s \in S$. By definition of $f^{-1}$, to compute $f^{-1}(f(s))$ I must find an element $\text{FOO} \in S$ such that $f(\text{FOO}) = f(s)$. But this is easy — just take $\text{FOO} = s$. Thus, $f^{-1}(f(s)) = s$.

Going the other way, let $t \in T$. By definition of $f^{-1}$, to compute $f(f^{-1}(t))$ I find an element $s \in S$ such that $f(s) = t$. Then $f^{-1}(t) = s$, so

$$f(f^{-1}(t)) = f(s) = t.$$

Therefore, $f^{-1}$ really is the inverse of $f$.

($\leftarrow$) Suppose $f$ has an inverse $f^{-1} : T \to S$. I must show $f$ is injective and surjective.

To show that $f$ is surjective, take $t \in T$. Then $f(f^{-1}(t)) = t$, so I’ve found an element of $S$ — namely $f^{-1}(t)$ — which $f$ maps to $t$. Therefore, $f$ is surjective.

To show that $f$ is injective, suppose $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$. Then

$$f^{-1}(f(s_1)) = f^{-1}(f(s_2)), \quad \text{so} \quad s_1 = s_2.$$

Therefore, $f$ is injective. □

**Corollary.** The composite of bijective functions is bijective.

**Proof.** Since a function is bijective if and only if it has an inverse, the corollary follows from the fact that the composite of invertible functions is invertible. □

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**Example.** Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by 

$$f(x, y) = (y - x^3, y + 17).$$ 

Prove that $f$ is bijective by constructing an inverse for $f$ and proving that it works.

First, I do some scratch work to guess the inverse. I need $f^{-1}(a, b) = (u, v)$, where $u$ and $v$ are the unknown outputs of $f^{-1}$ in terms of $a$ and $b$. Now

$$f(f^{-1}(a, b)) = f(u, v)$$

$$(a, b) = f(u, v)$$

$$(a, b) = (v - u^3, v + 17)$$

Equating corresponding components, I get

$$v - u^3 = a, \quad v + 17 = b.$$

The second equation gives $v = b - 17$. Plugging this into the first equation, I get

$$(b - 17) - u^3 = a$$

$$b - a - 17 = u^3$$

$$\sqrt{b - a - 17} = u$$

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Based on this scratch work, I'll define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f^{-1}(a, b) = (\sqrt[b]{b-a-17}, b-17).$$

Note that I got this by working backwards; I still need to verify that $f$ and $f^{-1}$ are inverses.

$$f^{-1}(f(x, y)) = f^{-1}(y-x^3, y+17)$$
$$= (\sqrt[3]{(y+17)} - (y-x^3) - 17, (y+17) - 17)$$
$$= (\sqrt[3]{x^3}, y)$$
$$= (x, y)$$

$$f(f^{-1}(a, b)) = f(\sqrt[b]{b-a-17}, b-17)$$
$$= ((b-17) - [\sqrt[b]{b-a-17}]^3, (b-17) + 17)$$
$$= (b-17 - (b-a-17), b)$$
$$= (a, b)$$

This proves that $f$ and $f^{-1}$ are inverses, so $f$ is bijective. □

**Example.** (a) Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither injective nor surjective.

(b) Prove that $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$ is not injective, but it is surjective.

(c) Prove that $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$ is injective and surjective.

(a) It is not injective, since $f(-3) = 9$ and $f(3) = 9$: Different inputs may give the same output.

It is not surjective, since there is no $x \in \mathbb{R}$ such that $f(x) = -1$. □

(b) It is not injective, since $f(-3) = 9$ and $f(3) = 9$: Different inputs may give the same output.

It is surjective, since if $y \geq 0$, $\sqrt{y}$ is defined, and

$$f(\sqrt{y}) = (\sqrt{y})^2 = y. □$$

(c) It is injective, since if $f(a) = f(b)$, then $a^2 = b^2$. But in this case, $a, b \geq 0$, so $a = b$ by taking square roots.

It is surjective, since if $y \geq 0$, then $\sqrt{y}$ is defined, and

$$f(\sqrt{y}) = (\sqrt{y})^2 = y. □$$

Notice that in this example, the same “rule” — $f(x) = x^2$ — was used, but whether the function was injective or surjective changed. *The domain and codomain are part of the definition of a function.* □

**Example.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \frac{x+1}{x}.$$  

Prove that $f$ is injective.

Suppose $a, b \in \mathbb{R}$ and $f(a) = f(b)$. I must prove that $a = b$.

$f(a) = f(b)$ means that

$$\frac{a+1}{a} = \frac{b+1}{b}.$$  

Clearing denominators and doing some algebra, I get

$$b(a+1) = a(b+1), \quad ba + b = ab + a, \quad a = b.$$
Therefore, $f$ is injective. 

**Example.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = 3x^7 + 5x^3 + 13x + 8.$$ 

Prove that $f$ is injective.

It would probably be difficult to prove this directly. Instead, I’ll use the following fact:

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, and that $f'(x) > 0$ for all $x$ or $f'(x) < 0$ for all $x$. Then $f$ is injective.

In this case, note that, since even powers are nonnegative,

$$f'(x) = 21x^6 + 15x^2 + 13 > 0.$$ 

Since the derivative is always positive, $f$ is always increasing, and hence $f$ is injective. 

Here’s a proof of the result I used in the last example.

**Proposition.** Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, and that $f'(x) > 0$ for all $x$ or $f'(x) < 0$ for all $x$. Then $f$ is injective.

**Proof.** Suppose that $f$ is differentiable and always increasing. Suppose that $f(a) = f(b)$. I want to prove that $a = b$.

Suppose on the contrary that $a \neq b$. There’s no harm in assuming $a < b$ (otherwise, switch them). By the Mean Value Theorem, there is a number $c$ such that $a < c < b$ and

$$f(b) - f(a) = f'(c).$$

Since $f(a) = f(b)$ and $f'(x) > 0$ for all $x$,

$$0 = f'(c) > 0$$

This contradiction proves that $a = b$. Therefore, $f$ is injective.

The same proof works with minor changes if $f'(x) < 0$ for all $x$. 

**Example.** Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 5x - 7$.

(a) Prove directly that $f$ is injective and surjective.

(b) Prove that $f$ is injective and surjective by showing that $f$ has an inverse $f^{-1}$.

(a) Suppose $f(a) = f(b)$. Then $5a - 7 = 5b - 7$, so $5a = 5b$, and hence $a = b$. Therefore, $f$ is injective.

Suppose $y \in \mathbb{R}$. I must find $x$ such that $f(x) = y$. I want $5x - 7 = y$. Working backwards, I find that $x = \frac{1}{5}(y + 7)$. Verify that it works:

$$f \left( \frac{1}{5}(y + 7) \right) = 5 \cdot \frac{1}{5}(y + 7) - 7 = (y + 7) - 7 = y.$$
This proves that $f$ is surjective. □

(b) Define $f^{-1}(x) = \frac{1}{5}(x + 7)$. I’ll prove that this is the inverse of $f$:

$$f(f^{-1}(x)) = f \left( \frac{1}{5}(x + 7) \right) = 5 \cdot \frac{1}{5}(x + 7) - 7 = (x + 7) - 7 = x,$$

$$f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{1}{5}((5x - 7) + 7) = \frac{1}{5} \cdot 5x = x.$$

Therefore, $f^{-1}$ is the inverse of $f$. Since $f$ is invertible, it’s injective and surjective. □

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**Example.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2(x - 1)$. Show that $f$ is not injective, but that $f$ is surjective. $f$ is not injective, since $f(0) = 0$ and $f(1) = 0$.

The graph suggests that $f$ is surjective. To say that every $y \in \mathbb{R}$ is an output of $f$ means graphically that every horizontal line crosses the graph at least once (whereas injectivity means that every horizontal line crosses that graph at most once).

![Graph of f(x) = x^2(x - 1)](image)

To prove that $f$ is surjective, take $y \in \mathbb{R}$. I must find $x \in \mathbb{R}$ such that $f(x) = y$, i.e. such that $x^2(x - 1) = y$.

The problem is that finding $x$ in terms of $y$ involves solving a cubic equation. This is possible, but it’s easy to change the example to produce a function where solving algebraically is impossible in principle.

Instead, I’ll proceed indirectly.

$$\lim_{x \to +\infty} x^2(x - 1) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} x^2(x - 1) = -\infty.$$

It follows from the definition of these infinite limits that there are numbers $a$ and $b$ such that

$$f(a) < y \quad \text{and} \quad f(b) > y.$$

But $f$ is continuous — it’s a polynomial — so by the Intermediate Value Theorem, there is a point $c$ such that $a < c < b$ and $f(c) = y$. This proves that $f$ is surjective.

Note, however, that I haven’t found $c$; I’ve merely shown that such a value $c$ must exist. □

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**Example.** Define

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}.$$

Prove that $f$ is surjective, but not injective.
Let $y \in \mathbb{R}$. If $y < 0$, then $y - 1 < -1 < 0$, so

$$f(y - 1) = (y - 1) + 1 = y.$$ If $y \geq 0$, then $\sqrt{y}$ is defined and $\sqrt{y} \geq 0$, so

$$f(\sqrt{y}) = (\sqrt{y})^2 = y.$$ This proves that $f$ is surjective.

However,

$$f \left( -\frac{3}{4} \right) = \frac{3}{4} + 1 = \frac{1}{4} \quad \text{and} \quad f \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^2 = \frac{1}{4}.$$ Hence, $f$ is not injective. □

**Example.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (4x - 3y, 2x + y).$$

(a) Show that $f$ is injective and surjective directly, using the definitions.

(b) Show that $f$ is injective and surjective by constructing an inverse $f^{-1}$.

(a) First, I’ll show that $f$ is injective. Suppose $f(a, b) = f(c, d)$. I want to show that $(a, b) = (c, d)$.

$f(a, b) = f(c, d)$ means

$$(4a - 3b, 2a + b) = (4c - 3d, 2c + d).$$

Equate corresponding components:

$$4a - 3b = 4c - 3d, \quad 2a + b = 2c + d.$$ Rewrite the equations:

$$4(a - c) = 3(b - d), \quad 2(a - c) = -(b - d).$$

The second of these equations gives $b - d = -2(a - c)$. Substitute this into the first equation:

$$4(a - c) = 3 \cdot (-2)(a - c)$$

$$4(a - c) = -6(a - c)$$

$$10(a - c) = 0$$

$$a - c = 0$$

$$a = c$$
Plugging this into $b - d = -2(a - c)$ gives $b - d = 0$, so $b = d$. Therefore, $(a, b) = (c, d)$, and $f$ is injective.

To show $f$ is surjective, I take a point $(a, b) \in \mathbb{R}^2$, the codomain. I must find $(x, y)$ such that $f(x, y) = (a, b)$.

I want 

$$(4x - 3y, 2x + y) = (a, b).$$

I’ll work backwards from this equation. Equating corresponding components gives

$$4x - 3y = a, \quad 2x + y = b.$$

The second equation gives $y = b - 2x$, so plugging this into the first equation yields

$$4x - 3(b - 2x) = a, \quad 10x - 3b = a, \quad x = 0.1a + 0.3b.$$

Plugging this back into $y = b - 2x$ gives

$$y = b - 2(0.1a + 0.3b) = -0.2a + 0.4b.$$

Now check that this works:

$$f(0.1a + 0.3b, -0.2a + 0.4b) = (4(0.1a + 0.3b) - 3(-0.2a + 0.4b), 2(0.1a + 0.3b) + (-0.2a + 0.4b)) = (a, b).$$

Therefore, $f$ is surjective. ☐

(b) I actually did the work of constructing the inverse in showing that $f$ was surjective: I showed that if $f(x, y) = (a, b)$, that

$$(x, y) = (0.1a + 0.3b, -0.2a + 0.4b), \quad \text{or} \quad f(0.1a + 0.3b, -0.2a + 0.4b) = (a, b).$$

But the second equation implies that if $f^{-1}$ exists, it should be defined by

$$f^{-1}(a, b) = (0.1a + 0.3b, -0.2a + 0.4b).$$

Now I showed above that

$$f(f^{-1}(a, b)) = f(0.1a + 0.3b, -0.2a + 0.4b) = (a, b).$$

For the other direction,

$$f^{-1}(f(x, y)) = f^{-1}(4x - 3y, 2x + y) = (0.1(4x - 3y) + 0.3(2x + y), -0.2(4x - 3y) + 0.4(2x + y)) = (x, y).$$

This proves that $f^{-1}$, as defined above, really is the inverse of $f$. Hence, $f$ is injective and surjective. ☐

In linear algebra, you learn more efficient ways to show that functions like the one above are bijective.

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**Example.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (2x + 4y, -x - 2y).$$

Prove that $f$ is neither injective nor surjective.

$f(0, 0) = (0, 0)$ and $f(2, -1) = (0, 0)$. 

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Therefore, $f$ is not injective.

To prove $f$ is not surjective, I must find a point $(a, b) \in \mathbb{R}^2$ which is not an output of $f$. I'll show that $(1, 1)$ is not an output of $f$. Suppose on the contrary that $f(x, y) = (1, 1)$. Then

$$(2x + 4y, -x - 2y) = (1, 1).$$

This gives two equations:

$$2x + 4y = 1, \quad -x - 2y = 1.$$

Multiply the second equation by $-2$ to obtain $2x + 4y = 2$. Now I have $2x + 4y = 1$ and $2x + 4y = -2$, so $1 = -2$, a contradiction.

Therefore, there is no such $(x, y)$, and $f$ is not surjective. □

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (e^x + y, y^3).$$

Is $f$ injective? Is $f$ surjective?

First, I'll show that $f$ is injective. Suppose $f(a, b) = f(c, d)$. I have to show that $(a, b) = (c, d)$.

$$f(a, b) = f(c, d)$$

$$(e^a + b, b^3) = (e^c + d, d^3)$$

Equating the second components, I get $b^3 = d^3$. By taking cube roots, I get $b = d$. Equating the first components, I get $e^a + b = e^c + d$. But $b = d$, so subtracting $b = d$ I get $e^a = e^c$. Now taking the log of both sides gives $a = c$. Thus, $(a, b) = (c, d)$, and $f$ is injective.

I'll show that $f$ is not surjective by showing that there is no input $(x, y)$ which gives $(-1, 0)$ as an output. Suppose on the contrary that $f(x, y) = (-1, 0)$. Then

$$f(x, y) = (-1, 0)$$

$$(e^x + y, y^3) = (-1, 0)$$

Equating the second components gives $y^3 = 0$, so $y = 0$. Equating the first components gives $e^x + y = -1$. But $y = 0$, so I get $e^x = -1$. This is impossible, since $e^x$ is always positive. Therefore, $f$ is not surjective. □