11-28-2018

Inequalities

We've already seen examples of proofs of inequalities as examples of various proof techniques. In this section, we'll discuss assorted inequalities and the heuristics involved in proving them. The subject of inequalities is vast, so our discussion will barely scratch the surface.

Here are a couple of basic rules which I'll use constantly.

1. You can add a number to both (or all sides) of an inequality.

2. You can multiply an inequality by a nonzero number — but if the number you multiply by is negative, the inequality is reversed.

Example. Prove that $x^4 + x^2y + 4y^2 \geq 5x^2y$.

The $x^4 + \cdots + 4y^2$ looks like it came from $(x^2 \pm 2y)^2$. I know that even powers are always ≥ 0. I'll start with $(x^2 - 2y)^2 \geq 0$ and see if I can get the desired inequality:

$$\begin{aligned} (x^2 - 2y)^2 &\geq 0 \\ x^4 - 4x^2y + 4y^2 &\geq 0 \\ x^4 + x^2y + 4y^2 &\geq 5x^2y \end{aligned} \qquad \square$$

Example. If $a, b > 0$, then $ab > 0$. And if $a = b$ and $c = d$, then $ac = bd$. Is it true that if $a > b$ and $c > d$, then $ac > bd$?

The statement is false. For example, $2 > 1$ and $-1 > -2$, but $2 \cdot (-1) \not> 1 \cdot (-2)$.

This result shows that you have to be careful in the rules you use to work with inequalities. Some "rules" which look obvious aren't correct. $\square$

In fact, the false result in the example can be "fixed" by placing additional assumptions on a, b, c, and d. To prove the correct result, I'll have to use *very* basic facts about inequalities involving real numbers.

Here are some *axioms* for the standard order relation on $\mathbb{R}$. Everything is defined in terms of a subset $\mathbb{R}^+$, the positive real numbers.

1. For every real number x, either $x \in \mathbb{R}^+$, $x = 0$, or $-x \in \mathbb{R}^+$.

2. The sum of positive real numbers is a positive real number.

3. The product of positive real numbers is a positive real number.

The order relation $>$ is defined in terms of $\mathbb{R}^+$. Think about a statement like "$7 > 3$". Another way to say this is: You add a *positive* number (namely 4) to 3 to get 7.

Definition. If $x, y \in \mathbb{R}$, $x > y$ means that $x = y + p$, where $p \in \mathbb{R}^+$.

As usual, $x < y$ means $y > x$, $x \geq y$ means $x > y$ or $x = y$, and $x \leq y$ means $y \geq x$.

Here is a "fixed" version of the incorrect rule in the last example. The proof illustrates a standard approach in inequality proofs involving the basic axioms: *Convert inequality statements to equations and work with the equations.*

Lemma. Suppose a, b, c, and d are positive real numbers, $a > b$, and $c > d$. Then $ac > bd$.

Proof. Suppose $a, b, c, d > 0$, $a > b$, and $c > d$. Write

$$a = b + p \quad \text{and} \quad c = d + q, \quad \text{where} \quad p, q \in \mathbb{R}^+.$$

Then

$$ac = (b + p)(d + q) = bd + pd + bq + pq.$$

pd, bq, and pq are positive, because each is the product of positive numbers. Hence, $pd + bq + pq$ is positive. The equation above therefore shows that $ac > bd$. □

Lemma. If $a > b$, then $a + c > b + c$.

Proof. Since $a > b$, I know that $a = b + p$, where $p \in \mathbb{R}^+$. Hence,

$$a + c = (b + c) + p.$$

But since $p \in \mathbb{R}^+$, this means that $a + c > b + c$. □

Plainly, the same proof works if addition is replaced with subtraction.

You can often prove an inequality by transforming or substituting in a known inequality. Unfortunately, there are infinitely many inequalities you can start with! You have to rely on your knowledge of mathematics, as well as common sense: If you're trying to prove an inequality in calculus, for example, it's natural to think of all the "calculus inequalities" you know.

Example. (**Using a known trig inequality**) Prove that for all $x \in \mathbb{R}$, $1 \le \dfrac{1}{2}(\sin x + 3) \le 2$.

The "$\cdot \le \cdot \le \cdot$" form of the inequality and the presence of the $\sin x$ in the middle remind me of $-1 \le \sin x \le 1$, so I'll start with that and do some algebra:

$$\begin{aligned} -1 \le\ & \sin x \le 1 \\ 2 \le\ & \sin x + 3 \le 4 \\ 1 \le\ & \frac{1}{2}(\sin x + 3) \le 2 \end{aligned} \quad □$$

Example. (**Using an integral inequality**) From calculus, you know that if f and g are integrable functions and $f(x) \ge g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

Use this inequality to prove that

$$0.1 \ge \int_0^{0.1} \frac{x^4}{x^4 + 1}\,dx.$$

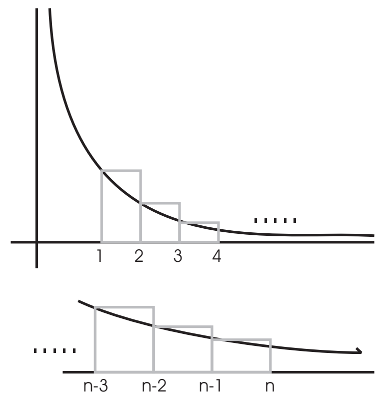
I have

$$\begin{aligned} 1 &\ge 0 \\ x^4 + 1 &\ge x^4 \\ 1 &\ge \frac{x^4}{x^4 + 1} \end{aligned}$$

Applying the integral inequality, I get

$$\int_0^{0.1} 1\,dx \ge \int_0^{0.1} \frac{x^4}{x^4 + 1}\,dx \quad \text{so} \quad 0.1 \ge \int_0^{0.1} \frac{x^4}{x^4 + 1}\,dx. \quad □$$

You can often "see" that an inequality is true by drawing a picture. For example, draw the graph of $y = \frac{1}{x}$ for $1 \le x \le n$, where n is an integer greater than 1.



Divide the interval $[1, n]$ up into n equal pieces, and build a rectangle on each piece, using the left-hand endpoints of each subinterval to get the heights. As you can see from the picture, the rectangles all lie above the curve, so the sum of the rectangle areas will be greater than the area under the curve.

The first rectangle has base 1 and height 1, so its area is 1. The second rectangle has base 1 and height $\frac{1}{2}$, so its area is $\frac{1}{2}$. Continuing in this way, the last rectangle has base 1 and height $\frac{1}{n-1}$, so its area is $\frac{1}{n-1}$.

The area under the curve is $\int_1^n \frac{dx}{x}$.

Therefore,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \ge \int_1^n \frac{dx}{x} = \ln n.$$

This inequality is correct (and by the way, you can use it to see that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges). But the argument I gave is not a completely rigorous proof.

I'm assuming that the picture accurately represents the situation. To *prove* that this is the case takes some work. For example, I'd need to prove that each rectangle really *does* lie above the curve. This would involve noting that $y' = -\frac{1}{x^2} < 0$ shows that the graph is decreasing, then using this to prove that the left-hand endpoints give the maximum value of $y = \frac{1}{x}$ on each subinterval.

Pictures can help you *see* or *remember* that something is true, and sometimes a picture or a *heuristic argument* is useful in teaching — to avoid obscuring the idea with technicalities. But you should never confuse a picture with a rigorous proof!

Example. (**Using the Mean Value Theorem**) Prove that for all $x > 0$, $e^x > x + 1$.

Let $f(x) = e^x - x - 1$. Take $x > 0$ and apply the Mean Value Theorem to f on the interval $[0, x]$. The Mean Value Theorem implies that there is a number c such that $0 < c < x$ and

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Now $f'(c) = e^c - 1$, and $c > 0$, so $f'(c) = e^c - 1 > 1 - 1 = 0$. Thus,

$$\frac{f(x) - f(0)}{x - 0} > 0$$
$$\frac{f(x)}{x} > 0$$
$$f(x) > 0$$

Therefore, $e^x - x - 1 > 0$, so $e^x > x + 1$. □

In some cases, you can use a known inequality to prove other inequalities. Here are some well-known inequalities.

The Triangle Inequality. Let $x, y \in \mathbb{R}$. Then

$$|x| + |y| \geq |x + y|.$$

The name "Triangle Inequality" comes from the corresponding inequality when x and y are vectors. In that case, it says that the sum of the lengths of two sides of a triangle ($|x| + |y|$) is greater than or equal to the length of the third side ($|x + y|$).

The Arithmetic-Geometric Mean Inequality. Let $x, y \in \mathbb{R}$, and suppose that $x, y \geq 0$. Then

$$\frac{x + y}{2} \geq \sqrt{xy}.$$

If x and y are two nonnegative numbers, their **arithmetic mean** ("average") is $\dfrac{x + y}{2}$ and their **geometric mean** is $\sqrt{xy}$ — hence the name of this inequality.

The Cauchy-Schwarz Inequality. If $x_1, \ldots, x_n, y_1, \ldots y_n \in \mathbb{R}$, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

You may have seen this inequality in a vector calculus course or a linear algebra course. Let

$$x = (x_1, \ldots, x_n) \quad \text{and} \quad y = (y_1, \ldots, y_n).$$

Then the vector form of the inequality is

$$(x \cdot y)^2 \leq \|x\|^2 \|y\|^2.$$

(The product on the left side is the dot product of the two vectors.)

Example. (**Using the Triangle Inequality**) Prove that if a and b are real numbers, then

$$|a - b| \geq ||a| - |b||.$$

Apply the Triangle Inequality with $x = a - b$ and $y = b$:

$$|a - b| + |b| \geq |(a - b) + b|$$
$$|a - b| + |b| \geq |a|$$
$$|a - b| \geq |a| - |b|$$

Apply the Triangle Inequality with $x = b - a$ and $y = a$:

$$\begin{aligned} |b-a| + |a| &\ge |(b-a) + a| \\ |b-a| + |a| &\ge |b| \\ |b-a| &\ge |b| - |a| \\ |a-b| &\ge |b| - |a| \end{aligned}$$

The last step follows from the fact that $|a - b| = |b - a|$.
Now

$$||a| - |b|| = \begin{cases} |a| - |b| & \text{if } |a| - |b| \ge 0 \\ -(|a| - |b|) = |b| - |a| & \text{if } |a| - |b| < 0 \end{cases}.$$

I've show that in both of these two cases, $|a - b| \ge ||a| - |b||$. Therefore, $|a - b| \ge ||a| - |b||$ for all $a, b \in \mathbb{R}$. $\square$

Example. (**Using the Cauchy-Schwarz Inequality**) Prove that if $a_1, \ldots a_n > 0$, then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \ge n^2.$$

Apply the Cauchy-Schwarz Inequality with

$$x_1 = \sqrt{a_1}, \ldots x_n = \sqrt{a_n}, \quad y_1 = \frac{1}{\sqrt{a_1}}, \ldots y_n = \frac{1}{\sqrt{a_n}}.$$

I get

$$\left(\sum_{k=1}^{n} \sqrt{a_k} \cdot \frac{1}{\sqrt{a_k}}\right)^2 \le \left(\sum_{k=1}^{n} (\sqrt{a_k})^2\right)\left(\sum_{k=1}^{n} \left(\frac{1}{\sqrt{a_k}}\right)^2\right).$$

This simplifies to

$$\left(\sum_{k=1}^{n} 1\right)^2 \le \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right).$$

But $\displaystyle\sum_{k=1}^{n} 1 = n$, so

$$n^2 \le \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right).$$

As in this example, the trick to applying known inequalities is figuring out what substitutions to make. $\square$
