Limits at Infinity

In this section, I’ll discuss proofs for limits of the form \( \lim_{x \to \infty} f(x) \). They are like \( \varepsilon-\delta \) proofs, though the setup and algebra are a little different.

Recall that \( \lim_{x \to c} f(x) = L \) means that for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( |x - c| > \delta \), then \( \varepsilon > |f(x) - L| \).

**Definition.** \( \lim_{x \to \infty} f(x) = L \) means that for every \( \varepsilon > 0 \), there is an \( M > 0 \) such that if \( x > M \), then \( \varepsilon > |f(x) - L| \).

In other words, I can make \( f(x) \) as close to \( L \) as I please by making \( x \) sufficiently large.

**Remarks.** Limits at infinity often occur as limits of sequences, such as

\[
\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots
\]

In this case, \( \lim_{n \to \infty} \frac{1}{n} = 0 \). I won’t make a distinction between the limit at infinity of a sequence and the limit at infinity of a function; the proofs you do are essentially the same in both cases.

There is a similar definition for \( \lim_{x \to -\infty} f(x) = L \), and the proofs are similar as well. I’ll stick to \( \lim_{x \to \infty} f(x) \) here.

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**Example.** Prove that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

As with \( \varepsilon-\delta \) proofs, I do some scratch work, working backwards from what I want. Then I write the “real proof” in the forward direction.

**Scratch work.** I want

\[
\varepsilon > \left| \frac{1}{n} - 0 \right| = \left| \frac{1}{n} \right| = \frac{1}{n}.
\]

I want to drop the absolute values, so I’ll assume \( n > 0 \). Rearranging the inequality, I get \( n > \frac{1}{\varepsilon} \).

Here’s the real proof. Let \( \varepsilon > 0 \). Set \( M = \frac{1}{\varepsilon} \). Since \( \varepsilon > 0 \), I have \( M = \frac{1}{\varepsilon} > 0 \). Suppose \( n > M \). Then \( n > M > 0 \), and

\[
\begin{align*}
n > M &= \frac{1}{\varepsilon} \\
\varepsilon &> \frac{1}{n} \\
\varepsilon &> \left| \frac{1}{n} \right| \\
\varepsilon &> \left| \frac{1}{n} - 0 \right|
\end{align*}
\]

This proves that \( \lim_{n \to \infty} \frac{1}{n} = 0 \). \( \square \)
Example. Prove that \( \lim_{x \to \infty} \frac{6x + 1}{2x + 1} = 3 \).

Scratch work. I want
\[
\epsilon > \left| \frac{6x + 1}{2x + 1} - 3 \right| = \left| \frac{6x + 1 - 3(2x + 1)}{2x + 1} \right| = \left| \frac{-2}{2x + 1} \right| = \left| \frac{2}{2x + 1} \right| = \frac{2}{2x + 1}.
\]

In order to drop the absolute values, I need to assume \( x > 0 \).
Rearrange the inequality:
\[
\epsilon > \frac{2}{2x + 1}
\]
\[
(2x + 1)\epsilon > 2
\]
\[
2x\epsilon + \epsilon > 2
\]
\[
2x\epsilon > 2 - \epsilon
\]
\[
x > \frac{2 - \epsilon}{2\epsilon}
\]

Here’s the real proof. Let \( \epsilon > 0 \). Set \( M = \max \left( 0, \frac{2 - \epsilon}{2\epsilon} \right) \). If \( x > M \), then \( x > 0 \) and \( x > \frac{2 - \epsilon}{2\epsilon} \). So
\[
x > \frac{2 - \epsilon}{2\epsilon}
\]
\[
2x\epsilon > 2 - \epsilon
\]
\[
2x\epsilon + \epsilon > 2
\]
\[
\epsilon(2x + 1) > 2
\]
\[
\epsilon > \frac{2}{2x + 1}
\]
\[
\epsilon > \left| \frac{2}{2x + 1} \right|
\]
\[
\epsilon > \left| \frac{-2}{2x + 1} \right|
\]
\[
\epsilon > \left| \frac{6x + 1 - 3(2x + 1)}{2x + 1} \right|
\]
\[
\epsilon > \left| \frac{6x + 1}{2x + 1} - 3 \right|
\]

Therefore,
\[
\lim_{x \to \infty} \frac{6x + 1}{2x + 1} = 3. \quad \Box
\]

Note that the expression \( \frac{2 - \epsilon}{2\epsilon} \) would be negative if \( \epsilon > 2 \). So I took \( M \) to be the max of 0 and \( \frac{2 - \epsilon}{2\epsilon} \) to ensure that if \( x > M \), then \( x \) would be positive. Now you actually need \( 2x + 1 \) to be positive in order to put on the absolute values, and \( 2x + 1 > 0 \) if \( x > -\frac{1}{2} \). It isn’t hard to prove that \( \frac{2 - \epsilon}{2\epsilon} > -\frac{1}{2} \), so in fact I don’t need to take the max with 0 — provided that I’m willing to prove that \( \frac{2 - \epsilon}{2\epsilon} > -\frac{1}{2} \). I decided to take the easy way out!
Example. Prove that \( \lim_{n \to \infty} (-1)^n \) is undefined.

I’ll use proof by contradiction. Suppose that

\[
\lim_{n \to \infty} (-1)^n = L.
\]

Taking \( \epsilon = \frac{1}{2} \) in the definition, I can find \( M \) such that if \( n > M \), then \( \frac{1}{2} > |(-1)^n - L| \).

Choose \( p \) to be an even number greater than \( M \). Then

\[
\frac{1}{2} > |(-1)^p - L| = |1 - L|.
\]

This says that the distance from \( L \) to 1 is less than \( \frac{1}{2} \), so

\[
\frac{1}{2} < L < \frac{3}{2}.
\]

Choose \( q \) to be an odd number greater than \( M \). Then

\[
\frac{1}{2} > |(-1)^q - L| = |-1 - L|.
\]

This says that the distance from \( L \) to \(-1\) is less than \( \frac{1}{2} \), so

\[
-\frac{3}{2} < L < -\frac{1}{2}.
\]

This is a contradiction, since \( L \) can’t be in \( \left( \frac{1}{2}, \frac{3}{2} \right) \) and in \( \left( -\frac{3}{2}, -\frac{1}{2} \right) \) at the same time.

Hence, \( \lim_{n \to \infty} (-1)^n \) is undefined. ☐