Limits

The definition of a limit involves both universal and existential quantifiers. Let \( f \) be a function from the real numbers to the real numbers, and let \( c \) be a real number. Assume that \( f \) is defined on a open interval containing \( c \). The statement \( \lim_{x \to c} f(x) = L \) means:

For every \( \epsilon > 0 \), there is a \( \delta > 0 \), such that if \( \delta > |x - c| > 0 \), then \( \epsilon > |f(x) - L| \).

Think of \( \delta \) as a thermostat, \( f(x) \) as the actual temperature in a room, and \( L \) as the ideal temperature. Someone challenges you to make the actual temperature \( f(x) \) fall within a certain tolerance \( \epsilon \) of the ideal temperature \( L \). You must do that by setting your \( \delta \)-thermostat appropriately (so that \( x \) is sufficiently close to \( c \)).

Moreover, note that it says “for every \( \epsilon > 0 \)”. It’s isn’t enough for you to say what you’d do if you were challenged with \( \epsilon = 0.1 \) or \( \epsilon = 0.000004 \). You must prove that you can meet the challenge no matter what \( \epsilon \) you’re challenged with.

Finally, note the stipulation “\( |x - c| > 0 \)”. This implies that \( x \neq c \), since \( x = c \) gives \( |x - c| = 0 \). Thus, the conclusion \( \epsilon > |f(x) - L| \)” must hold only for \( x \)’s close to \( c \), but not necessarily for \( x = c \). (It may hold for \( x = c \), but it doesn’t have to.)

What does this mean? It’s a precise way of saying that the value of the limit of \( f(x) \) as \( x \) approaches \( c \) does not depend on what \( f(x) \) does at \( x = c \) — over even whether \( f(c) \) is defined.

For example, consider the functions whose graphs are shown below.

In both cases,

\[
\lim_{x \to 3} f(x) = 4.
\]

In the first case, \( f(3) = 2 \): The value of the function at \( x = 3 \) is different from the value of the limit.

In the second case, \( f(3) \) is undefined.

The fact that \( \lim_{x \to 3} f(x) \neq f(3) \) means that \( f \) is not continuous at \( x = 3 \).

Example. Use the \( \epsilon \)-\( \delta \) definition of the limit to prove that

\[
\lim_{x \to 2} (5x + 4) = 14.
\]

In this case, \( c = 2 \), \( f(x) = 5x + 4 \), and \( L = 14 \). So here is what I need to prove.

Suppose \( \epsilon > 0 \). I must find a \( \delta > 0 \) such that if \( \delta > |x - 2| > 0 \), then \( \epsilon > |(5x + 4) - 14| \).

Note that at this point \( \epsilon \) is fixed — given — but all you can assume is that it’s some positive number. Since it is given, however, I can use it in finding an appropriate \( \delta \).
I’ll show how to find $\delta$ by working backwards; then I’ll write the proof “forwards”, the way you should write it.

I want

$$\epsilon > |(5x + 4) - 14|, \text{ or } \epsilon > |5x - 10|, \text{ or } \frac{\epsilon}{5} > |x - 2|.$$ 

It looks like I should set $\delta = \frac{\epsilon}{5}$.

All of this has been on “scratch paper”; now here’s the real proof.

Suppose $\epsilon > 0$. Let $\delta = \frac{\epsilon}{5}$. If $\delta > |x - 2| > 0$, then

$$\frac{\epsilon}{5} > |x - 2|, \text{ so } \epsilon > |5x - 10|, \text{ or } \epsilon > |(5x + 4) - 14|.$$ 

Thus, if $\delta = \frac{\epsilon}{5}$ and $\delta > |x - 2| > 0$, then $\epsilon > |(5x + 4) - 14|$. This proves that $\lim_{x \to 2} (5x + 4) = 14$. □

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**Example.** Let

$$f(x) = \begin{cases} 
3x + 4 & \text{if } x < 1 \\
9 - 2x & \text{if } x \geq 1
\end{cases}.$$ 

Use the $\epsilon$-$\delta$ definition of the limit to prove that

$$\lim_{x \to 1} f(x) = 7.$$ 

Let $\epsilon > 0$. I must find $\delta > 0$ such that if $\delta > |x - 1| > 0$, then $\epsilon > |f(x) - 7|$.

Here’s my scratch work. First, for $x < 1$,

$$\epsilon > |f(x) - 7|, \epsilon > |(3x + 4) - 7|, \epsilon > |3x - 3|, \frac{\epsilon}{3} > |x - 1|.$$ 

It looks like I should take $\delta = \frac{\epsilon}{3}$.

For $x > 1$,

$$\epsilon > |f(x) - 7|, \epsilon > |(9 - 2x) - 7|, \epsilon > |2 - 2x| = |2x - 2|, \frac{\epsilon}{2} > |x - 1|.$$ 

It looks like I should take $\delta = \frac{\epsilon}{2}$.

In order to ensure that both the $x < 1$ and $x > 1$ requirements are satisfied, I’ll take $\delta$ to be the smaller of the two: $\delta = \frac{\epsilon}{3}$.

Now here’s the proof written out correctly.

Suppose $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$ and assume that $\delta > |x - 1| > 0$.

If $x < 1$, then

$$\frac{\epsilon}{3} > |x - 1|, \text{ so } \epsilon > |3x - 3| = |(3x + 4) - 7| = |f(x) - 7|.$$ 

Now consider the case $x > 1$. Since $\frac{\epsilon}{3} > |x - 1|$, and since $\frac{\epsilon}{2} > \frac{\epsilon}{3}$, I have $\frac{\epsilon}{2} > |x - 1|$. Therefore,

$$\epsilon > |2x - 2| = |2 - 2x| = |(9 - 2x) - 7| = |f(x) - 7|.$$ 

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Example. Use the $\epsilon$-$\delta$ definition of the limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$ 

Let $\epsilon > 0$. I want to find $\delta > 0$ such that if $\delta > |x - 2| > 0$, then $\epsilon > |x^2 - 4|$.

I start out as usual with my scratch work:

$$\epsilon > |x^2 - 4| = |x - 2||x + 2|.$$ 

Now I have a problem. I can use $\delta$ to control $|x - 2|$, but what do I do about $|x + 2|$?

The idea is this: Since I have complete control over $\delta$, I can assume $\delta \leq 1$. When I finally set $\delta$, I can make it smaller if necessary to ensure that this condition is met.

Now if $\delta \leq 1$, then $|x - 2| < 1$, so $1 < x < 3$, and $3 < x + 2 < 5$. In particular, the biggest $|x + 2|$ could be is 5. So now

$$\epsilon > |x - 2||x + 2| \quad \text{becomes} \quad \epsilon > |x - 2| \cdot 5, \quad \text{so} \quad \frac{\epsilon}{5} > |x - 2|.$$ 

This inequality suggests that I set $\delta = \frac{\epsilon}{5}$ — but then I remember that I needed to assume $\delta \leq 1$. I can meet both of these conditions by setting $\delta$ to the smaller of 1 and $\frac{\epsilon}{5}$: that is, $\delta = \min \left( 1, \frac{\epsilon}{5} \right)$.

That was scratchwork; now here’s the real proof.

Let $\epsilon > 0$. Set $\delta = \min \left( 1, \frac{\epsilon}{5} \right)$. Suppose $\delta > |x - 2| > 0$.

Since $\delta \leq 1$, I have

$$1 > |x - 2|$$

$$1 < x < 3$$

$$3 < x + 2 < 5$$

Therefore, $5 > |x + 2|$.

Now $\delta \leq \frac{\epsilon}{5}$, so $\frac{\epsilon}{5} > |x - 2|$.

Now multiply the inequalities $5 > |x + 2|$ and $\frac{\epsilon}{5} > |x - 2|$:

$$\epsilon = \frac{\epsilon}{5} \cdot 5 > |x - 2||x + 2| = |x^2 - 4|.$$ 

Thus, if $\delta = \min \left( 1, \frac{\epsilon}{5} \right)$ and $\delta > |x - 2| > 0$, then $\epsilon > |x^2 - 4|$. This proves that $\lim_{x \to 2} x^2 = 4$. \qed

Example. Prove that $\lim_{x \to 2} \frac{x^2 + 11}{x + 3} = 3$.

Let $\epsilon > 0$. I must find $\delta$ such that if $\delta > |x - 2| > 0$, then $\epsilon > \left| \frac{x^2 + 11}{x + 3} - 3 \right|$.

I’ll start with some scratchwork.

$$\left| \frac{x^2 + 11}{x + 3} - 3 \right| = \left| \frac{x^2 + 11 - 3(x + 3)}{x + 3} \right| = \left| \frac{x^2 - 3x + 2}{x + 3} \right| = \left| \frac{(x - 2)(x - 1)}{x + 3} \right| = \left| x - 2 \right| \left| \frac{x - 1}{x + 3} \right|.$$
I can use $\delta$ to control $|x - 2|$ directly. I need to control the size of $\frac{|x - 1|}{x + 3}$. It’s important to think of this as $|x - 1| \cdot \frac{1}{x + 3}$, not as $|x - 1|$ and $|x + 3|$!

Assume $1 \geq \delta$. Then $1 > |x - 2|$, so $1 < x < 3$.

For $x - 1$, $0 < x - 1 < 2$, so $|x - 1| < 2$.

For $\frac{1}{x + 3}$, $4 < x + 3 < 6$, so $\frac{1}{4} > \frac{1}{x + 3} > \frac{1}{6}$, and $\frac{1}{x + 3} < \frac{1}{4}$.

Since all the number involved are positive, I can multiply the inequalities to obtain

$$2 \cdot \frac{1}{4} > |x - 1| \cdot \frac{1}{x + 3}, \quad \text{or} \quad \frac{1}{2} > |x - 1| \cdot \frac{1}{x + 3}.$$

Thus, I’ll get $\epsilon > |x - 2| \cdot \frac{|x - 1|}{x + 3}$ if I have $\epsilon > |x - 2| \cdot \frac{1}{2}$, or $2\epsilon > |x - 2|$. Here’s the proof.

Let $\epsilon > 0$. Set $\delta = \min(2\epsilon, 1)$. Suppose $\delta > |x - 2| > 0$.

Since $1 \geq \delta$, $1 > |x - 2|$, and $1 < x < 3$.

First, $0 < x - 1 < 2$, so $|x - 1| < 2$.

Next, $\frac{1}{x + 3}$, $4 < x + 3 < 6$, so $\frac{1}{4} > \frac{1}{x + 3} > \frac{1}{6}$, and $\frac{1}{x + 3} < \frac{1}{4}$.

Hence,

$$2 \cdot \frac{1}{4} > |x - 1| \cdot \frac{1}{x + 3}, \quad \text{or} \quad \frac{1}{2} > |x - 1| \cdot \frac{1}{x + 3}.$$

In addition, $2\epsilon \geq \delta > |x - 2|$. Therefore,

$$\epsilon > |x - 2| \cdot \frac{1}{2} > |x - 2| \cdot |x - 1| \cdot \frac{1}{x + 3} = \frac{x^2 + 11}{x + 3} - 3.$$

This proves that $\lim_{x \to 2} \frac{x^2 + 11}{x + 3} = 3$. ∎