9-30-2018

Proof by Cases

You can sometimes prove a statement by:

1. Dividing the situation into **cases** which exhaust all the possibilities; and

2. Showing that the statement follows in all cases.

It's important to cover all the possibilities. And don't confuse this with *trying examples*; an example is not a proof.

Note that there are usually many ways to divide a situation into cases. For example, if I know that x is a real number and I'm proving something about x, here are some ways I could take cases:

(a) $x > 0$, $x = 0$, or $x < 0$.

(b) $|x| > 1$ or $|x| \leq 1$.

(c) $x \geq \pi$ or $x < \pi$

(d) x is rational or x is irrational.

There are an *infinite* number of ways to divide up the real numbers to take cases; how you do it depends on what you're trying to prove. In general, you should try to use a small number of cases — and in particular, you should see if you can give a proof without taking cases at all!

I'll begin with a logic proof. In this situation, your cases are usually P and $\sim P$, where P is a statement.

Example. Premises: $\begin{cases} A \to (B \wedge \sim D) \\ C \to A \\ C \vee \sim D \end{cases}$

Prove: $\sim D$.

I can divide the situation into two cases: Either C is true, or $\sim C$ is true. These exhaust the possibilities, by the Law of the Excluded Middle. I'll assume each in turn and show that I can derive $\sim D$.

1.	$A \to (B \wedge \sim D)$	Premise
2.	$C \to A$	Premise
3.	$C \vee \sim D$	Premise
4.	C	Premise - Case 1
5.	A	Modus ponens (2,4)
6.	$B \wedge \sim D$	Modus ponens (1,5)
7.	$\sim D$	Decomposing a conjunction (6)
8.	$\sim C$	Premise - Case 2
9.	$\sim D$	Disjunctive syllogism (3,8)
10.	$\sim D$	Proof by cases (4,7,8,9)

Since both of my cases led to the conclusion $\sim D$, and since my cases exhausted the possibilities, I've proved $\sim D$.

In logic proofs, cases of the form P and $\sim P$ where P is some statement will cover all possibilities, since one of P or $\sim P$ must be true. So these are the natural cases to take in logic proofs.

How did I know to use C and $\sim C$ rather than (say) B and $\sim B$? I looked at my premises and noticed that I could do something with each of those assumptions: C could be used for modus ponens, and $\sim C$

could be used for disjunctive syllogism. As with many logic proofs, it was a matter of looking ahead or working backward.

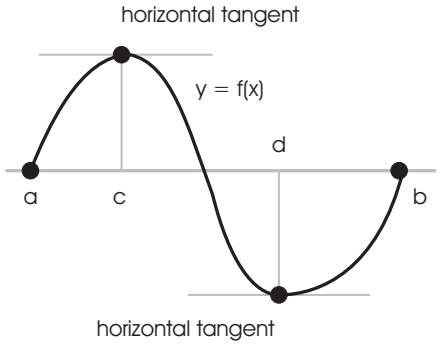
Note: You may use the premises for the proof in either case, but you may not use a statement derived for one case in the other case.

For example, in the first case, I derived the statement A at line 5. I may not use A anywhere in the second case. □

Example. In calculus, you learned **Rolle's theorem**. Here's the statement:

Let f be a function which is continuous on the interval $a \leq x \leq b$ and is differentiable on the interval $a < x < b$. Suppose $f(a) = f(b) = 0$. Then there is a real number c such that $a < c < b$ and $f'(c) = 0$.

In other words (to put it roughly), between two roots there must be a horizontal tangent.

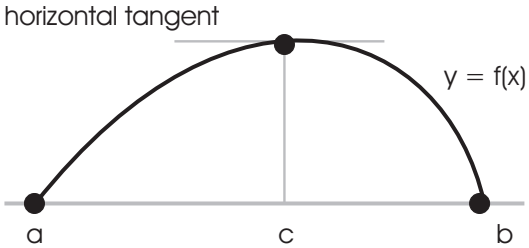


Prove Rolle's Theorem by taking cases.

There are three cases: f is never positive or negative on the interval $a \leq x \leq b$, f is positive somewhere on the interval $a \leq x \leq b$, or f is negative somewhere on the interval $a \leq x \leq b$.

Suppose first that f is never positive or negative on the interval $a \leq x \leq b$. Then $f = 0$, a constant function, and $f'(x) = 0$ for all x in the interval $a < x < b$.

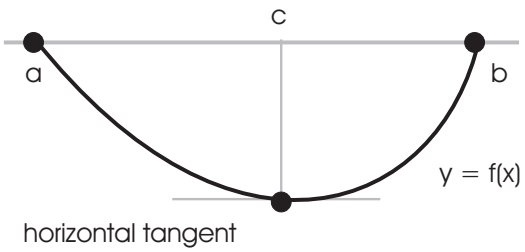
Suppose that f is positive at some point of the interval $a \leq x \leq b$. A continuous function on a closed interval attains a maximum value on the interval; since I already know f is positive *somewhere*, the maximum value of f must be positive. Since f is 0 at the endpoints, it must attain the maximum value at some point c in the open interval $a < x < b$.



Since $a < c < b$, f is differentiable at c. But at a point where a differentiable function attains a maximum, the derivative is 0. Therefore, $f'(c) = 0$.

Suppose that f is negative at some point of the interval $a \leq x \leq b$. A continuous function on a closed interval attains a minimum value on the interval; since I already know f is negative *somewhere*, the minimum value of f must be negative. Since f is 0 at the endpoints, it must attain the minimum value at some point

c in the open interval $a < x < b$.



Since $a < c < b$, f is differentiable at c. But at a point where a differentiable function attains a minimum, the derivative is 0. Therefore, $f'(c) = 0$.

Since the three cases exhaust all the possibilities, this proves that $f'(c) = 0$ for some c in the interval $a < x < b$. □

Many problems involving divisibility of integers use the **Division Algorithm**. It is a consequence of the **Well-Ordering Axiom** for the positive integers, which is also the basis for **mathematical induction**.

Theorem. (Division Algorithm) Let m and n be integers, where $n > 0$. Then there are unique integers q and r such that

$$m = nq + r, \quad \text{where} \quad 0 \leq r < n.$$

("q" stands for "quotient" and "r" stands for "remainder".)

I won't give a proof of this, but here are some examples which show how it's used.

Example. Apply the Division Algorithm to:

(a) Divide 31 by 8.

(b) Divide -31 by 8.

(c) Divide an integer m by 2.

(a) Let $m = 31$ and $n = 8$. Then I have

$$31 = 8 \cdot 3 + 7.$$

In this case, $q = 3$ and $r = 7$. Note that $0 \leq 7 < 8$ holds — when you divide, the remainder should be nonnegative, and less than the number you divided by. □

(b) (a) Let $m = -31$ and $n = 8$. Then I have

$$-31 = 8 \cdot (-4) + 1.$$

In this case, $q = -4$ and $r = 1$. Again, $0 \leq 1 < 8$ holds. Note that if I wrote "$-31 = 8 \cdot (-3) + (-7)$", the equation is true, but the numbers **aren't** the ones produced by the Division Algorithm — r is not allowed to be negative. □

(c) Take m to be an integer, and let $n = 2$. Then

$$m = 2q + r, \quad \text{where} \quad 0 \leq r < 2.$$

Now since r is an integer and $0 \leq r < 2$, I must have $r = 0$ or $r = 1$. Thus, if $m \in \mathbb{Z}$, then

$$m = 2q \quad \text{or} \quad m = 2q + 1.$$

Of course, the first case occurs when m is even, and the second case occurs when m is odd. *If a problem involves odd or even integers, you might consider taking cases in this way.* □

A similar situation occurs when n is any positive integer. For example, if $m \in \mathbb{Z}$ and $n = 5$, then

$$m = 5q + r, \quad \text{where} \quad 0 \leq r < 5.$$

The condition $0 \leq r < 5$ means $r = 0$, $r = 1$, $r = 2$, $r = 3$, or $r = 4$. So if $m \in \mathbb{Z}$, the possibilities are

$$m = 5q, \quad m = 5q + 1, \quad m = 5q + 2, \quad m = 5q + 3, \quad \text{or} \quad m = 5q + 4.$$

If a problem involves divisibility by 5 *you might consider taking cases in this way.*
(When I discuss **modular arithmetic**, there will be an easier way to deal with these cases.)

Example. Prove that if n is an integer, then $3n^2 + n + 14$ is even.

Let $n \in \mathbb{Z}$. I'll consider two cases: n is even and n is odd.

Case 1. n is even.

Since n is even, I can write $n = 2k$, where $k \in \mathbb{Z}$. Then

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k)^2 + 2k + 14 \\ &= 12k^2 + 2k + 14 \\ &= 2(6k^2 + k + 7) \end{aligned}$$

Since $6k^2 + k + 7$ is an integer, $3n^2 + n + 14$ is even if n is even.

Case 2. n is odd.

Since n is odd, I can write $n = 2k + 1$, where $k \in \mathbb{Z}$. Then

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k + 1)^2 + (2k + 1) + 14 \\ &= 3(4k^2 + 4k + 1) + (2k + 1) + 14 \\ &= 12k^2 + 12k + 3 + 2k + 1 + 14 \\ &= 12k^2 + 14k + 18 \\ &= 2(6k^2 + 7k + 9) \end{aligned}$$

Since $6k^2 + 7k + 9$ is an integer, $3n^2 + n + 14$ is even if n is odd.

Since in both cases $3n^2 + n + 14$ is even, it follows that if n is an integer, then $3n^2 + n + 14$ is even. □

Example. Prove that if n is any integer which is not divisible by 5, then n^2 leaves a remainder of 1 or 4 when it is divided by 5.

Let n be an integer which is not divisible by 5. I want to show that n^2 leaves a remainder of 1 or 4 when it is divided by 5.

Since n is not divisible by 5, it leaves a remainder of 1, 2, 3, or 4 when it is divided by 5. These four cases exhaust all the possibilities.

If n leaves a remainder of 1 when it's divided by 5, then $n = 5k + 1$ for some integer k. So

$$n^2 = (5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1.$$

Therefore, n^2 leaves a remainder of 1 when it's divided by 5.
If n leaves a remainder of 2 when it's divided by 5, then $n = 5k + 2$ for some integer k. So

$$n^2 = (5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4.$$

Therefore, n^2 leaves a remainder of 4 when it's divided by 5.
If n leaves a remainder of 3 when it's divided by 5, then $n = 5k + 3$ for some integer k. So

$$n^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 25k^2 + 30k + 5 + 4 = 5(5k^2 + 6k + 1) + 4.$$

Therefore, n^2 leaves a remainder of 4 when it's divided by 5.
If n leaves a remainder of 4 when it's divided by 5, then $n = 5k + 4$ for some integer k. So

$$n^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 25k^2 + 40k + 15 + 1 = 5(5k^2 + 8k + 3) + 1.$$

Therefore, n^2 leaves a remainder of 1 when it's divided by 5.
I've exhausted all the cases. This proves that if n is any integer which is not divisible by 5, then n^2 leaves a remainder of 1 or 4 when it is divided by 5.
This is not the best way to write this kind of proof, since the algebra can be a bit annoying. Proofs like this one can be written more easily using **modular arithmetic**. □

Example. Prove that for all $x \in \mathbb{R}$,

$$-5 \leq |x + 2| - |x - 3| \leq 5.$$

You often think of taking cases in dealing with absolute values. I have

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 > 0 \\ -(x + 2) & \text{if } x + 2 \leq 0 \end{cases}.$$

Now $x + 2 > 0$ means $x > -2$, and $x + 2 \leq 0$ means $x \leq -2$. So

$$|x + 2| = \begin{cases} x + 2 & \text{if } x > -2 \\ -(x + 2) & \text{if } x \leq -2 \end{cases}.$$

In the same way,

$$|x - 3| = \begin{cases} x - 3 & \text{if } x > 3 \\ -(x - 3) & \text{if } x \leq 3 \end{cases}.$$

Given the way the functions are broken apart, I'll consider the cases $x \leq -2$, $-2 < x \leq 3$, and $x > 3$. Notice that all real numbers are in one of the three cases.

Case 1. $x \leq -2$. In this case,

$$|x + 2| - |x - 3| = -(x + 2) - [-(x - 3)] = -5.$$

Therefore, $-5 \leq |x + 2| - |x - 3| \leq 5$ holds in this case.

Case 2. $-2 < x \leq 3$. In this case,

$$|x + 2| - |x - 3| = (x + 2) - [-(x - 3)] = 2x - 1.$$

I have to do some additional work to see whether the target inequality holds. I have

$$\begin{aligned} -2 &< x \leq 3 \\ -4 &< 2x \leq 6 \\ -5 &< 2x - 1 \leq 5 \end{aligned}$$

Therefore, $-5 \leq |x+2| - |x-3| \leq 5$ holds in this case.

Case 3. $x > 3$. In this case,

$$|x+2| - |x-3| = (x+2) - (x-3) = 5.$$

Therefore, $-5 \leq |x+2| - |x-3| \leq 5$ holds in this case.

Since $-5 \leq |x+2| - |x-3| \leq 5$ holds all three cases, it is true for all $x \in \mathbb{R}$. □
