Proof by Cases

You can sometimes prove a statement by:

1. Dividing the situation into cases which exhaust all the possibilities; and
2. Showing that the statement follows in all cases.

It’s important to cover all the possibilities. And don’t confuse this with trying examples; an example is not a proof.

Note that there are usually many ways to divide a situation into cases. For example, if I know that \( x \) is a real number and I’m proving something about \( x \), here are some ways I could take cases:

(a) \( x > 0 \), \( x = 0 \), or \( x < 0 \).
(b) \( |x| > 1 \) or \( |x| \leq 1 \).
(c) \( x \geq \pi \) or \( x < \pi \)
(d) \( x \) is rational or \( x \) is irrational.

There are an infinite number of ways to divide up the real numbers to take cases; how you do it depends on what you’re trying to prove. In general, you should try to use a small number of cases — and in particular, you should see if you can give a proof without taking cases at all!

I’ll begin with a logic proof. In this situation, your cases are usually \( P \) and \( \sim P \), where \( P \) is a statement.

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Example. Premises: \[
\begin{align*}
A &\to (B \land \sim D) \\
C &\to A \\
C \lor \sim D
\end{align*}
\]

Prove: \( \sim D \).

I can divide the situation into two cases: Either \( C \) is true, or \( \sim C \) is true. These exhaust the possibilities, by the Law of the Excluded Middle. I’ll assume each in turn and show that I can derive \( \sim D \).

1. \( A \to (B \land \sim D) \) Premise
2. \( C \to A \) Premise
3. \( C \lor \sim D \) Premise
4. \( C \) Premise - Case 1
5. \( A \) Modus ponens (2,4)
6. \( B \land \sim D \) Modus ponens (1,5)
7. \( \sim D \) Decomposing a conjunction (6)
8. \( \sim C \) Premise - Case 2
9. \( \sim D \) Disjunctive syllogism (3,8)
10. \( \sim D \) Proof by cases (4,7,8,9)

Since both of my cases led to the conclusion \( \sim D \), and since my cases exhausted the possibilities, I’ve proved \( \sim D \).

In logic proofs, cases of the form \( P \) and \( \sim P \) where \( P \) is some statement will cover all possibilities, since one of \( P \) or \( \sim P \) must be true. So these are the natural cases to take in logic proofs.

How did I know to use \( C \) and \( \sim C \) rather than (say) \( B \) and \( \sim B \)? I looked at my premises and noticed that I could do something with each of those assumptions: \( C \) could be used for modus ponens, and \( \sim C \)
could be used for disjunctive syllogism. As with many logic proofs, it was a matter of looking ahead or working backward.

**Note:** You may use the premises for the proof in either case, but you may not use a statement derived for one case in the other case.

For example, in the first case, I derived the statement $A$ at line 5. I may not use $A$ anywhere in the second case.

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**Example.** In calculus, you learned Rolle's theorem. Here's the statement:

Let $f$ be a function which is continuous on the interval $a \leq x \leq b$ and is differentiable on the interval $a < x < b$. Suppose $f(a) = f(b) = 0$. Then there is a real number $c$ such that $a < c < b$ and $f'(c) = 0$.

In other words (to put it roughly), between two roots there must be a horizontal tangent.

![Graph of a function with horizontal tangents at points a, c, and b.](image)

Prove Rolle’s Theorem by taking cases.

There are three cases: $f$ is never positive or negative on the interval $a \leq x \leq b$, $f$ is positive somewhere on the interval $a \leq x \leq b$, or $f$ is negative somewhere on the interval $a \leq x \leq b$.

Suppose first that $f$ is never positive or negative on the interval $a \leq x \leq b$. Then $f = 0$, a constant function, and $f'(x) = 0$ for all $x$ in the interval $a < x < b$.

Suppose that $f$ is positive at some point of the interval $a \leq x \leq b$. A continuous function on a closed interval attains a maximum value on the interval; since I already know $f$ is positive somewhere, the maximum value of $f$ must be positive. Since $f$ is 0 at the endpoints, it must attain the maximum value at some point $c$ in the open interval $a < x < b$.

![Graph of a function with horizontal tangent at point c.](image)

Since $a < c < b$, $f$ is differentiable at $c$. But at a point where a differentiable function attains a maximum, the derivative is 0. Therefore, $f'(c) = 0$.

Suppose that $f$ is negative at some point of the interval $a \leq x \leq b$. A continuous function on a closed interval attains a minimum value on the interval; since I already know $f$ is negative somewhere, the minimum value of $f$ must be negative. Since $f$ is 0 at the endpoints, it must attain the minimum value at some point...
c in the open interval \( a < x < b \).

Since \( a < c < b \), \( f \) is differentiable at \( c \). But at a point where a differentiable function attains a minimum, the derivative is 0. Therefore, \( f'(c) = 0 \).

Since the three cases exhaust all the possibilities, this proves that \( f'(c) = 0 \) for some \( c \) in the interval \( a < x < b \).

Many problems involving divisibility of integers use the Division Algorithm. It is a consequence of the Well-Ordering Axiom for the positive integers, which is also the basis for mathematical induction.

**Theorem. (Division Algorithm)** Let \( m \) and \( n \) be integers, where \( n > 0 \). Then there are unique integers \( q \) and \( r \) such that

\[
m = nq + r, \quad \text{where} \quad 0 \leq r < n.
\]

(“\( q \)” stands for “quotient” and “\( r \)” stands for “remainder”.)

I won’t give a proof of this, but here are some examples which show how it’s used.

**Example.** Apply the Division Algorithm to:

(a) Divide 31 by 8.
(b) Divide \(-31\) by 8.
(c) Divide an integer \( m \) by 2.

(a) Let \( m = 31 \) and \( n = 8 \). Then I have

\[
31 = 8 \cdot 3 + 7.
\]

In this case, \( q = 3 \) and \( r = 7 \). Note that \( 0 \leq 7 < 8 \) holds — when you divide, the remainder should be nonnegative, and less than the number you divided by.

(b) (a) Let \( m = -31 \) and \( n = 8 \). Then I have

\[
-31 = 8 \cdot (-4) + 1.
\]

In this case, \( q = -4 \) and \( r = 1 \). Again, \( 0 \leq 1 < 8 \) holds. Note that if I wrote \( "-31 = 8 \cdot (-3) + (-7)" \), the equation is true, but the numbers aren’t the ones produced by the Division Algorithm — \( r \) is not allowed to be negative.

(c) Take \( m \) to be an integer, and let \( n = 2 \). Then

\[
m = 2q + r, \quad \text{where} \quad 0 \leq r < 2.
\]

Now since \( r \) is an integer and \( 0 \leq r < 2 \), I must have \( r = 0 \) or \( r = 1 \). Thus, if \( m \in \mathbb{Z} \), then

\[
m = 2q \quad \text{or} \quad m = 2q + 1.
\]
Of course, the first case occurs when \( m \) is even, and the second case occurs when \( m \) is odd. If a problem involves odd or even integers, you might consider taking cases in this way.

A similar situation occurs when \( n \) is any positive integer. For example, if \( m \in \mathbb{Z} \) and \( n = 5 \), then

\[
m = 5q + r, \quad \text{where} \quad 0 \leq r < 5.
\]

The condition \( 0 \leq r < 5 \) means \( r = 0, r = 1, r = 2, r = 3, \) or \( r = 4 \). So if \( m \in \mathbb{Z} \), the possibilities are

\[
m = 5q, \quad m = 5q + 1, \quad m = 5q + 2, \quad m = 5q + 3, \quad \text{or} \quad m = 5q + 4.
\]

If a problem involves divisibility by 5 you might consider taking cases in this way.

(When I discuss modular arithmetic, there will be an easier way to deal with these cases.)

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**Example.** Prove that if \( n \) is an integer, then \( 3n^2 + n + 14 \) is even.

Let \( n \in \mathbb{Z} \). I’ll consider two cases: \( n \) is even and \( n \) is odd.

**Case 1.** \( n \) is even.

Since \( n \) is even, I can write \( n = 2k \), where \( k \in \mathbb{Z} \). Then

\[
3n^2 + n + 14 = 3(2k)^2 + 2k + 14
= 12k^2 + 2k + 14
= 2(6k^2 + k + 7)
\]

Since \( 6k^2 + k + 7 \) is an integer, \( 3n^2 + n + 14 \) is even if \( n \) is even.

**Case 2.** \( n \) is odd.

Since \( n \) is odd, I can write \( n = 2k + 1 \), where \( k \in \mathbb{Z} \). Then

\[
3n^2 + n + 14 = 3(2k + 1)^2 + (2k + 1) + 14
= 3(4k^2 + 4k + 1) + (2k + 1) + 14
= 12k^2 + 12k + 3 + 2k + 1 + 14
= 12k^2 + 14k + 18
= 2(6k^2 + 7k + 9)
\]

Since \( 6k^2 + 7k + 9 \) is an integer, \( 3n^2 + n + 14 \) is even if \( n \) is odd.

Since in both cases \( 3n^2 + n + 14 \) is even, it follows that if \( n \) is an integer, then \( 3n^2 + n + 14 \) is even.

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**Example.** Prove that if \( n \) is any integer which is not divisible by 5, then \( n^2 \) leaves a remainder of 1 or 4 when it is divided by 5.

Let \( n \) be an integer which is not divisible by 5. I want to show that \( n^2 \) leaves a remainder of 1 or 4 when it is divided by 5.

Since \( n \) is not divisible by 5, it leaves a remainder of 1, 2, 3, or 4 when it is divided by 5. These four cases exhaust all the possibilities.

If \( n \) leaves a remainder of 1 when it’s divided by 5, then \( n = 5k + 1 \) for some integer \( k \). So

\[
n^2 = (5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1.
\]
Therefore, \( n^2 \) leaves a remainder of 1 when it’s divided by 5.

If \( n \) leaves a remainder of 2 when it’s divided by 5, then \( n = 5k + 2 \) for some integer \( k \). So
\[
    n^2 = (5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4.
\]

Therefore, \( n^2 \) leaves a remainder of 4 when it’s divided by 5.

If \( n \) leaves a remainder of 3 when it’s divided by 5, then \( n = 5k + 3 \) for some integer \( k \). So
\[
    n^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 25k^2 + 30k + 5 + 4 = 5(5k^2 + 6k + 1) + 4.
\]

Therefore, \( n^2 \) leaves a remainder of 1 when it’s divided by 5.

If \( n \) leaves a remainder of 4 when it’s divided by 5, then \( n = 5k + 4 \) for some integer \( k \). So
\[
    n^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 25k^2 + 40k + 15 + 1 = 5(5k^2 + 8k + 3) + 1.
\]

Therefore, \( n^2 \) leaves a remainder of 4 when it’s divided by 5.

I’ve exhausted all the cases. This proves that if \( n \) is any integer which is not divisible by 5, then \( n^2 \) leaves a remainder of 1 or 4 when it is divided by 5.

This is not the best way to write this kind of proof, since the algebra can be a bit annoying. Proofs like this one can be written more easily using modular arithmetic. \( \square \)

**Example.** Prove that for all \( x \in \mathbb{R} \),
\[
    -5 \leq |x + 2| - |x - 3| \leq 5.
\]

You often think of taking cases in dealing with absolute values. I have
\[
    |x + 2| = \begin{cases} 
    x + 2 & \text{if } x + 2 > 0 \\
    -(x + 2) & \text{if } x + 2 \leq 0
    \end{cases}.
\]

Now \( x + 2 > 0 \) means \( x > -2 \), and \( x + 2 \leq 0 \) means \( x \leq -2 \). So
\[
    |x + 2| = \begin{cases} 
    x + 2 & \text{if } x > -2 \\
    -(x + 2) & \text{if } x \leq -2
    \end{cases}.
\]

In the same way,
\[
    |x - 3| = \begin{cases} 
    x - 3 & \text{if } x > 3 \\
    -(x - 3) & \text{if } x \leq 3
    \end{cases}.
\]

Given the way the functions are broken apart, I’ll consider the cases \( x \leq -2 \), \( -2 < x \leq 3 \), and \( x > 3 \). Notice that all real numbers are in one of the three cases.

**Case 1.** \( x \leq -2 \). In this case,
\[
    |x + 2| - |x - 3| = -(x + 2) - -(x - 3) = -5.
\]

Therefore, \( -5 \leq |x + 2| - |x - 3| \leq 5 \) holds in this case.

**Case 2.** \( -2 < x \leq 3 \). In this case,
\[
    |x + 2| - |x - 3| = (x + 2) - -(x - 3) = 2x - 1.
\]

I have to do some additional work to see whether the target inequality holds. I have
\[
    \begin{align*}
    -2 < x & \leq 3 \\
    -4 < 2x & \leq 6 \\
    -5 < 2x - 1 & \leq 5
    \end{align*}
\]

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Therefore, \(-5 \leq |x + 2| - |x - 3| \leq 5\) holds in this case.

**Case 3.** \(x > 3\). In this case,

\[ |x + 2| - |x - 3| = (x + 2) - (x - 3) = 5. \]

Therefore, \(-5 \leq |x + 2| - |x - 3| \leq 5\) holds in this case.

Since \(-5 \leq |x + 2| - |x - 3| \leq 5\) holds all three cases, it is true for all \(x \in \mathbb{R}\). \(\blacksquare\)