Set Algebra and Proofs Involving Sets

There are a lot of rules involving sets; you’ll probably become familiar with the most important ones simply by using them a lot. Usually you can check informally (for instance, by using a Venn diagram) whether a rule is correct; if necessary, you should be able to write a proof. In most cases, you can give a proof by going back to the definitions of set contructions in terms of elements.

Example. (Distributivity) Let $A$, $B$, and $C$ be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If $X$ and $Y$ are sets, $X = Y$ if and only if for all $x$, $x \in X$ if and only if $x \in Y$.

Let $x$ be an arbitrary element of the universe.

$$x \in A \cap (B \cup C) \iff x \in A \land x \in (B \cup C) \quad \text{Definition of } \cap$$
$$\iff x \in A \land (x \in B \lor x \in C) \quad \text{Definition of } \cup$$
$$\iff (x \in A \land x \in B) \lor (x \in A \land x \in C) \quad \text{Distributivity of } \land \text{ over } \lor$$
$$\iff (x \in A \land x \in C) \quad \text{Definition of } \cap$$
$$\iff x \in (A \cap B) \lor (A \cap C) \quad \text{Definition of } \cup$$

I’ve shown that

$$x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C).$$

By definition of set equality, this proves that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. \[\Box\]

The idea of the proof was to reduce everything to statements about elements. Then I used logical rules to manipulate the element statements.

Note: It’s also true that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example. (DeMorgan’s Law) Let $A$ and $B$ be sets. Prove that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$\[\Box\]

I’ll just prove the first statement; the second is similar. This proof will illustrate how you can work with complements. I’ll use the logical version of DeMorgan’s law to do the proof.

Let $x$ be an arbitrary element of the universe.

$$x \in \overline{A \cup B} \iff x \notin A \cup B \quad \text{Definition of union}$$
$$\iff x \notin A \land x \notin B \quad \text{Definition of } \notin$$
$$\iff \sim (x \in A) \land \sim (x \in B) \quad \text{Definition of } \land$$
$$\iff (x \notin A) \land (x \notin B) \quad \text{DeMorgan’s law}$$
$$\iff (x \notin A) \land (x \notin B) \quad \text{Definition of } \notin$$
$$\iff (x \notin \overline{A}) \land (x \notin \overline{B}) \quad \text{Definition of complement}$$
$$\iff x \in \overline{A} \cap \overline{B} \quad \text{Definition of union}$$

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. \[\Box\]
Example. Let $A$ and $B$ be sets. Prove that $A \cap B \subset A$.

This example will show how you prove a subset relationship.

By definition, if $X$ and $Y$ are sets, $X \subset Y$ if and only if for all $x$, if $x \in X$, then $x \in Y$.

Take an arbitrary element $x$. Suppose $x \in A \cap B$ (conditional proof). I want to show that $x \in A$.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, by definition of intersection. But $x \in A$ and $x \in B$ implies $x \in A$ (decomposing a conjunction), and this is what I wanted to show. Therefore, $A \cap B \subset A$.

By the way, you usually don’t write the logic out in such gory detail. The proof above could be shortened to:

$x \in A \cap B$ means that $x \in A$ and $x \in B$, so in particular $x \in A$. Therefore, $A \cap B \subset A$.

The “in particular” substitutes for decomposing the conjunction.

The procedure I’ve followed is so common that it’s worth pointing it out.

To prove a subset relationship (an inclusion) $X \subset Y$, take an arbitrary element of $X$ and prove that it must be in $Y$.

In the next example, I’ll need the following facts from logic. First, $P \lor \sim P$ is a tautology:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\sim P$</th>
<th>$P \lor \sim P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Also, $P \land (\text{a tautology}) \leftrightarrow P$:

<table>
<thead>
<tr>
<th>$P$</th>
<th>a tautology</th>
<th>$P \land (\text{a tautology})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

In effect, this means that I can drop tautologies from “and” statements. I’ll just call this “Dropping tautologies” in the proof below.

Example. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

$x \in (A - B) \cup (B - A) \leftrightarrow$
$x \in (A - B) \lor x \in (B - A) \leftrightarrow$
$(x \in A \land x \notin B) \lor (x \in B \land x \notin A) \leftrightarrow$
$[x \in A \lor (x \in B \land x \notin A)] \land [x \notin B \lor (x \in B \land x \notin A)] \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$[x \in A \lor (x \in B \land x \notin A)] \land [x \notin B \lor (x \in B \land x \notin A)] \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$[x \in A \lor (x \in B \land x \notin A)] \land [x \notin B \lor (x \in B \land x \notin A)] \leftrightarrow$
$(x \in A \lor x \notin B) \land (x \notin B \lor x \notin A) \leftrightarrow$
$x \in (A \cup B) \land \sim (x \in A \cap B) \leftrightarrow$

$x \in (A \cup B) \land \sim (x \in A \cap B) \leftrightarrow$

Therefore, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. □
Example. Let $A$ be a set. Prove that

$$A \cup \emptyset = A \quad \text{and} \quad A \cap \emptyset = \emptyset.$$ 

This example will show how you can deal with the empty set.

To prove $A \cup \emptyset = A$, let $x$ be an arbitrary element of the universe. First, by definition of $\cup$,

$$x \in A \cup \emptyset \iff (x \in A) \lor (x \in \emptyset).$$

I’ll show that $[(x \in A) \lor (x \in \emptyset)] \iff (x \in A)$. To prove $P \iff Q$, I must prove $P \rightarrow Q$ and $Q \rightarrow P$.

First, if $x \in A$, then $(x \in A) \lor (x \in \emptyset)$ (constructing a disjunction).

Next, suppose $(x \in A) \lor (x \in \emptyset)$. The second statement $x \in \emptyset$ is false for all $x$, by definition of $\emptyset$. But the $\lor$-statement is true by assumption, so $x \in A$ must be true by disjunctive syllogism. This proves that if $(x \in A) \lor (x \in \emptyset)$, then $x \in A$.

This completes my proof that $[(x \in A) \lor (x \in \emptyset)] \iff (x \in A)$. So

$$x \in A \cup \emptyset \iff (x \in A) \lor (x \in \emptyset) \quad \text{Definition of } \cup$$

$$\iff x \in A \quad \text{Proved above}$$

Therefore, $A \cup \emptyset = A$.

To prove that $A \cap \emptyset = \emptyset$, I must prove that for all $x$, $x \in A \cap \emptyset$ if and only if $x \in \emptyset$.

As usual, $x$ be an arbitrary element of the universe. To prove $x \in A \cap \emptyset$ if and only if $x \in \emptyset$, I must prove that the two implications

$$(x \in A \cap \emptyset) \rightarrow x \in \emptyset \quad \text{and} \quad x \in \emptyset \rightarrow (x \in A \cap \emptyset)$$

are true. I’ll do this by showing that, in each case, the antecedent (i.e. the “if” part of the statement) is false — since by basic logic, if $P$ is false, then $P \rightarrow Q$ is true.

For the first implication, consider the statement $x \in A \cap \emptyset$. By definition of intersection,

$$x \in A \cap \emptyset \iff (x \in A \land x \in \emptyset).$$

Now $x \in \emptyset$ is false, by definition of the empty set. Therefore, the conjunction $x \in A \land x \in \emptyset$ is also false. Hence, $x \in A \cap \emptyset$ is false.

It follows that the implication $x \in A \cap \emptyset \rightarrow x \in \emptyset$ is true, because the “if” part is false.

Likewise, the second implication $x \in \emptyset \rightarrow (x \in A \cap \emptyset)$ is true because $x \in \emptyset$ is false, by definition of the empty set.

Since both implications are true, $x \in A \cap \emptyset$ if and only if $x \in \emptyset$. And this in turn proves that $A \cap \emptyset = \emptyset$. $\Box$