Rational Approximation by Continued Fractions

- The convergents of a continued fraction expansion of x give the **best rational approximations** to x. Specifically, the only way a fraction can approximate x *better than* a convergent is if the fraction has a bigger denominator than the convergent.

The first lemma says that the denominators of convergents of continued fractions increase.

Lemma. Let a_0, a_1, a_2, ... be a sequence of integers, where $a_k > 0$ for $k \geq 1$. Define

$$p_0 = a_0, \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geq 2.$$

Then $q_{k+1} > q_k$ for $k > 0$.

Proof. Let $k > 0$. Note that q_{k-1} is a positive integer. So

$$q_{k+1} = a_{k+1} q_k + q_{k-1} > a_{k+1} q_k \geq 1 \cdot q_k = q_k,$$

where $a_{k+1} \geq 1$ because the a's are positive integers from a_1 on. □

The convergents of a continued fraction oscillate around the limiting value, and the convergents are always fractions in lowest terms. In fact, the convergents are the *best rational approximations* to the value of the continued fraction. I'll state the precise result without proof.

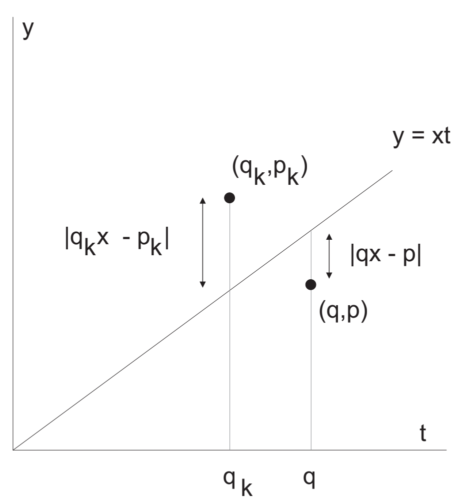
Theorem. Let x be irrational, and let $c_k = \dfrac{p_k}{q_k}$ be the k-th convergent in the continued fraction expansion of x. Suppose $p, q \in \mathbb{Z}$, $q > 0$, and

$$|qx - p| < |q_k x - p_k|.$$

Then $q \geq q_{k+1}$. □

Here's what the result means. Draw the line through the origin in the t-y plane with slope x. Plot the points (p, q) and (p_k, q_k).

The hypothesis $|qx - p| < |q_k x - p_k|$ says that the vertical distance from (q, p) to $y = xt$ is less than the vertical distance from (q_k, p_k) to $y = xt$.



The conclusion says that $q \geq q_{k+1}$. In fact, since $q_{k+1} > q_k$, $q > q_k$: The denominator of $\frac{p}{q}$ is bigger than that of $\frac{p_k}{q_k}$.

In other words, the only way the point (p, q) can be closer to the line is if its y-coordinate is bigger.

I can restate the theorem in the form of a corollary in which you can see the fractions in question approximating x.

Corollary. Let x be irrational, and let $c_k = \frac{p_k}{q_k}$ be the k-th convergent in the continued fraction expansion of x. Suppose $p, q \in \mathbb{Z}$, $q > 0$, and

$$\left|x - \frac{p}{q}\right| < \left|x - \frac{p_k}{q_k}\right|.$$

Then $q > q_k$.

Proof. Given the hypotheses of the corollary, suppose on the contrary that $q \leq q_k$. Since

$$\left|x - \frac{p}{q}\right| < \left|x - \frac{p_k}{q_k}\right|,$$

I can multiply the two inequalities to get

$$|qx - p| < |q_k x - p_k|.$$

Apply the theorem to obtain $q \geq q_{k+1}$. But then $q_k \geq q \geq q_{k+1}$, which contradicts the fact that the q's increase.

Therefore, $q > q_k$. □

This result says that the only way a rational number $\frac{p}{q}$ can approximate a continued fraction *better than* a convergent $\frac{p_k}{q_k}$ is if the fraction has a bigger denominator than the convergent.

Example. Here are the convergents for the continued fraction expansion for π:

a_k	p_k	q_k	c_k
3	3	1	3
7	22	7	$\frac{22}{7}$
15	333	106	$\frac{333}{106}$
1	355	113	$\frac{355}{113}$
292	103993	33102	$\frac{103993}{33102}$

$\frac{355}{113} \approx 3.141592920$, which is in error in the seventh place. The theorem says that a fraction $\frac{p}{q}$ can be closer to π than $\frac{355}{113}$ only if $q > 113$. □

The next result is sort of a converse to the previous two results. It says that if a rational number approximates an irrational number x "sufficiently well", then the rational number must be a convergent in the continued fraction expansion for x.

Theorem. Let x be irrational, and let $\frac{p}{q}$ be a rational number in lowest terms with $q > 0$. Suppose that

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent in the continued fraction expansion for x.

Proof. Since $q_k \geq k$ for $k \geq 0$, the q's form a strictly increasing sequence of positive integers. Therefore, for some k,

$$q_k \leq q < q_{k+1}.$$

Since $q < q_{k+1}$, the contrapositive of the preceding theorem gives

$$|q_k x - p_k| \leq |qx - p| = q\left|x - \frac{p}{q}\right| < q \cdot \frac{1}{2q^2} = \frac{1}{2q}.$$

Hence,

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{2qq_k}.$$

Now assume toward a contradiction that $\frac{p}{q}$ is *not* a convergent in the continued fraction expansion for x. In particular, $\frac{p}{q} \neq \frac{p_k}{q_k}$, so $qp_k \neq pq_k$, and hence $|qp_k - pq_k|$ is a positive integer.

Since $|qp_k - pq_k| \geq 1$,

$$\frac{1}{qq_k} \leq \frac{|qp_k - pq_k|}{qq_k} = \left|\frac{p_k}{q_k} - \frac{p}{q}\right| = \left|\frac{p_k}{q_k} - x + x - \frac{p}{q}\right| \leq \left|\frac{p_k}{q_k} - x\right| + \left|x - \frac{p}{q}\right| < \frac{1}{2qq_k} + \frac{1}{2q^2}.$$

(The second inequality comes from the Triangle Inequality: $|a + b| \leq |a| + |b|$.)

Subtracting $\frac{1}{2qq_k}$ from both sides, I get

$$\frac{1}{2qq_k} < \frac{1}{2q^2}, \quad \text{so} \quad q < q_k.$$

But I assumed $q_k \leq q$, so this is a contradiction.

Therefore, $\frac{p}{q}$ is a convergent in the continued fraction expansion for x. □

Example. Show that $\frac{355}{113}$ is the best rational approximation to π by a fraction having a denominator less than 1000.

Suppose that $\frac{p}{q}$ is a fraction in lowest terms that is a better approximation to π than $\frac{355}{113}$, and that $q < 1000$.

Since $\frac{p}{q}$ is a fraction is a better approximation to π than $\frac{355}{113}$,

$$\left|\pi - \frac{p}{q}\right| < \left|\pi - \frac{355}{113}\right|.$$

Since $q < 1000$,

$$2q^2 < 2000000, \quad \text{so} \quad \frac{1}{2q^2} > \frac{1}{2000000} = 5 \times 10^{-7}.$$

But

$$\left|\pi - \frac{355}{113}\right| = 2.66764\ldots \times 10^{-7}.$$

Thus,

$$\frac{1}{2q^2} > 5 \times 10^{-7} > \left|\pi - \frac{355}{113}\right| > \left|\pi - \frac{p}{q}\right|.$$

The hypotheses of the theorem are satisfied, so $\frac{p}{q}$ must be a convergent in the continued fraction expansion of π.

But the other convergents with denominators less than 1000 — 3, $\frac{22}{7}$, $\frac{333}{106}$ — with denominators less than 1000 are *poorer* approximations to π than $\frac{355}{113}$.

Hence, $\frac{355}{113}$ is the best rational approximation to π by a fraction having a denominator less than 1000. □

Example. (a) Compute the first 6 convergents $c_0, \ldots c_5$ of the continued fraction for $11^{1/3}$.

(b) Show that $\frac{278}{125}$ is the best rational approximation to $11^{1/3}$ having denominator less than 155.

(a)

x	a	p	q	c
2.223980090569315	2	2	1	2
4.46468254146245	4	9	4	$\frac{9}{4}$
2.152006823524719	2	20	9	$\frac{20}{9}$
6.578652042139306	6	129	58	$\frac{129}{58}$
1.728154274376962	1	149	67	$\frac{149}{67}$
1.373335342782462	1	278	125	$\frac{278}{125}$
2.6785570114363653	2	705	317	$\frac{705}{317}$

□

(b) Suppose that $\frac{p}{q}$ is a fraction in lowest terms which is a better approximation to $11^{1/3}$ than $\frac{278}{125}$, and also that $q < 155$.

Since $\frac{p}{q}$ is a better approximation to $11^{1/3}$ than $\frac{278}{125}$,

$$\left|11^{1/3} - \frac{p}{q}\right| < \left|11^{1/3} - \frac{278}{125}\right| = 1.99094\ldots \times 10^{-5}.$$

Since $q < 155$,

$$\begin{aligned} q &< 155 \\ q^2 &< 24025 \\ 2q^2 &< 48050 \\ \frac{1}{2q^2} &> \frac{1}{48050} = 2.08116\ldots \times 10^{-5} \end{aligned}$$

So I have

$$\frac{1}{2q^2} > 2.08116\ldots \times 10^{-5} > 1.99094\ldots \times 10^{-5} > \left|11^{1/3} - \frac{p}{q}\right|.$$

(The inequalities are approximate, but there is enough room between $2.08116\ldots \times 10^{-5}$ and $1.99094\ldots \times 10^{-5}$ that there is no problem.)

By the approximation theorem, $\frac{p}{q}$ is a convergent for $11^{1/3}$. But no convergent with $q < 155$ is a better approximation than $\frac{278}{125}$.

This contradiction shows that $\frac{278}{125}$ is the best rational approximation to $11^{1/3}$ having denominator less than 155. □