Binomial Coefficients

If \( n \) and \( k \) are integers, \( n \geq 0 \), and \( 0 \leq k \leq n \), then the binomial coefficient \( \binom{n}{k} \) (read \text{n-choose-k}) is defined by

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Why “n-choose-k”? Suppose you have \( n \) different objects. How many ways are there of choosing \( k \) of them (without worrying about the order of choice)?

The first object can be chosen in \( n \) ways. Then there are \( n-1 \) objects left, so the second can be chosen in \( n-1 \) ways. Then there are \( n-2 \) objects left, so the third can be chosen in \( n-2 \) ways. And so on. At the \( k \)th choice, you have \( n-k+1 \) objects to choose from. So the number of ways of choosing \( k \) objects in a particular order is

\[
n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{1 \cdot 2 \cdots n}{1 \cdot 2 \cdots (n-k)} = \frac{n!}{(n-k)!}.
\]

However, since I don’t care about the order in which I choose the \( k \) objects (only which \( k \) objects are chosen), I have to divide by the number of different orders in which I could have chosen the \( k \) objects. This is the number of permutations of \( k \) objects, which is \( k! \). Hence, the number of ways of choosing \( k \) objects from a set of \( n \), without regard for order, is

\[
\frac{1}{k!} \cdot \frac{n!}{(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\]

Example. Compute \( \binom{5}{3} \) and \( \binom{10}{10} \).

\[
\binom{5}{3} = \frac{5!}{3!2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2} = \frac{4 \cdot 5}{1 \cdot 2} = 10.
\]

\[
\binom{10}{10} = \frac{10!}{10!0!} = 1.
\]

Proposition. (Properties of binomial coefficients)

(a) \( \binom{n}{n} = \binom{n}{0} = 1 \).

(b) \( \binom{n}{k} = \binom{n}{n-k} \).

(c) (Pascal’s triangle) \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).
The last property has the following pictorial interpretation.

Make a triangle as shown by starting at the top and writing 1’s down the sides. Then fill in the middle of the triangle one row at a time, by adding the elements diagonally above the new element. For example, the leftmost 4 in the \( n = 4 \) row was obtained this way:

\[
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1 \\
\end{array}
\begin{array}{c}
\searrow \\
\nearrow \\
\searrow \\
\nearrow \\
\arrow \downarrow \\
\end{array}
\begin{array}{c}
4 \\
3 \\
2 \\
1 \\
\end{array}
\]

The formula above is simply an algebraic expression of this addition procedure.

**Proof.** You can check the formulas in (a) and (b) by writing out the binomial coefficients. Here’s the computation for one part of (a):

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = 1.
\]

And here’s the computation for (b):

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.
\]

The proof of (c) is also a computation, though it’s a little more involved:

\[
\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{(k-1)!(n-k)!(n-k+1)!} \left( \frac{1}{k} + \frac{1}{n-k+1} \right) = \frac{n!}{(k-1)!(n-k)!(n-k+1)!} \cdot \frac{n+1}{k(n-k+1)} = \frac{(n+1)!}{k!(n-k+1)!}.
\]

Of course, **binomial coefficients** get their name because they’re the coefficients in the expansion of a binomial:

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

Since the coefficients can be read off from Pascal’s triangle, you can use the triangle to write down binomial expansions.

**Example.** Use Pascal’s triangle to compute the binomial expansion of \((x + y)^4\).
Using the \( n = 4 \) row in the triangle, I get

\[
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \]

\[
\]

\[\text{Example.} \text{ Determine the coefficient of } x^{37}y^3 \text{ in the expansion of } (2x - 3y)^{40}. \]

The term containing \( x^{37}y^3 \) is

\[
\binom{40}{37}(2x)^{37}(-3y)^3 = \frac{40!}{37!3!}2^{37}(-3)^3x^{37}y^3 = -38 \cdot 39 \cdot 40 \cdot 2^{36} \cdot 3^2 \cdot x^{37}y^3.
\]

(I cancelled the 37! with the first 37 terms in 40!, then cancelled the 3! = 3 \cdot 2 with 2^{37} and 3^3.) The coefficient is \(-38 \cdot 39 \cdot 40 \cdot 2^{36} \cdot 3^2\), or \(-36663215228190720\) if you multiply it out. \( \square \)

\[\text{Example.} \text{ Prove that } 4^n < \binom{2n}{n} \text{ for } n \geq 1. \]

Some results involving binomial coefficients can be proven by choosing an appropriate binomial expansion. In this case, I notice that the “2\( n \)” in the binomial coefficient would come from expanding \((x + y)^{2n}\). But what should I choose for \( x \) and \( y \)?

After some trial and error, I find that this works:

\[
4^n = 2^{2n} = (1 + 1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k 1^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k}.
\]

The binomial coefficient \( \binom{2n}{n} \) is the middle term in this sum — but being the middle term, it is also the largest term in the sum. Look at Pascal’s Triangle to convince yourself that this is true:

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

There are \( 2n + 1 \) terms in the sum, and all of them are less than \( \binom{2n}{n} \). Thus,

\[
4^n = \sum_{k=0}^{2n} \binom{2n}{k} = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} + \cdots + \binom{2n}{2n} < \binom{2n}{0} + \binom{2n}{n} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n} = (2n + 1) \binom{2n}{n}
\]

Dividing both sides by \( 2n + 1 \), I get

\[
\frac{4^n}{2n + 1} < \binom{2n}{n}. \quad \square
\]

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